Discrete Mathematical Structures
CS 3233 Lecture Four

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Business

- Questions???
- Kallen (our TA) will be available by appointment after class on Tuesday and Thursday
  - Arrange appointment with her at class time.
- Read Section 1.4 by Thursday
- Recall: Homework due Thursday September 6
  - Section 1.2: 2, 6, 10, 12
  - Section 1.3: 10d, 10e, 14, 24c, 24d, 32a, 32b, 44

Section 1.3: Predicates

- Propositional functions
- Examples
  - p(x) ≡ x > 3
  - q(y) ≡ y has paid his tuition
  - r(z) ≡ z has wireless access on campus
- p(4) has a truth value
- What does r(x) → q(x) mean?

Predicates, Continued

- Each propositional function takes a fixed number of arguments
  - older(x,y) ≡ x is older than y
  - Here "older" is being used as a propositional function
- A propositional function p and a predicate p are the same thing
- A statement of the form $p(x_1, x_2, \ldots, x_n)$ is a proposition

Formulas Involving Predicates

- Let ϕ(y) ≡ q(y) → r(y)
  - We say ϕ is *a formula in* y (not in text)
- Then ϕ(Fred) ≡ q(Fred) → r(Fred)
  - This is an example of substitution of variables by values

Universe of Discourse and Quantifiers

- The *universe of discourse* or *domain* is the set of all possible values for variables
- We can refer to values in the universe either by using constant symbols (like "Fred") or by using quantifiers
- There are two quantifiers in standard predicate calculus: *for all* (∀) and *there exists* (∃)
- There are called the universal quantifier and the existential quantifier, respectively

Universal Quantifiers

- The universal quantification of p(x) is the following proposition:
  - "p(x) is true for all values of x in the universe of discourse"
  - Written $\forall x\ p(x)$ or $\forall x.p(x)$
- Similarly, if $\phi(x)$ is a formula in x, $\forall x.\phi(x)$ means the formula holds for all elements of the universe
  - What does $\forall x.(r(x) \rightarrow q(x))$ mean?
  - How is it different from $(\forall x.r(x)) \rightarrow (\forall x.q(x))$?

Universal Quantifiers

- Note that $\forall x.p(x) \equiv \forall y.p(y)$
- If the universe of discourse is {0, 1, 2}, then $\forall x.p(x) \equiv p(0) \wedge p(1) \wedge p(2)$
- Can you always rewrite $\forall x.p(x)$ this way?
  - What if the universe of discourse is infinite?
  - Logical statements are finite objects

Existential Quantifiers

- The existential quantification of p(x) is the following proposition:
  - "there exists an element x in the universe of discourse such that p(x) is true"
  - Written $\exists x\ p(x)$ or $\exists x.p(x)$
  - What does $\exists x.(r(x) \wedge \neg q(x))$ mean?
- If the universe of discourse is {0,1,2}, then
  - $\exists x.p(x) \equiv p(0) \vee p(1) \vee p(2)$

Scope of Quantifiers

- $(\exists x.x>3) \wedge (\exists x.x<1)$
- The scope of a quantifier is the formula following the dot

Negations of Quantified Formulas

- De Morgan's Laws for Quantifiers

  $\neg\exists x.p(x) \equiv \forall x.\neg p(x)$

  $\neg\forall x.p(x) \equiv \exists x.\neg p(x)$

Predicate Formulas as Specifications

- For the domain of integers, is $\forall x.(x>3)$ true or false?
  - How about $\exists x.(x>3)$?
  - Note: ">" is the predicate symbol here
  - X>3 is another way of writing >(x,3)
- What does $\forall x\ \forall y.(x+y = y+x)$ mean?
  - Is it true?
  - What is the predicate here?
  - In logic, "+" is called a *function symbol* (term not introduced in the text)