

Solutions: Mock Exam for Final Discrete Mathematical Structures CS3233

December, 2007

The final exam is cumulative. This mock exam covers material starting with that presented in lecture 11/13/2007, which was not previously tested. This includes material in sections 5.1, 5.2, 5.3 and 6.1. Students should refer to midterms 1 and 2, and to the corresponding mock exams and homeworks, for assistance in studying material previously tested.

Where numerical answers are requested, do not fully evaluate your expressions. Simplify your answers so that they are expressed in terms of factorials and exponents, but not in terms of $C(n, r)$ or $P(n, r)$. This is how you will answer this sort of question on the final.

1. How many r -letter words can be constructed using an alphabet of size n ?

Solution: n^r

2. How many ways are there to order a set of size n ?

Solution: $n!$

3. How many ways are there to order a set of size n if the first r places are always occupied by the same r elements?

Solution: $(n - r)!$

4. How many ways are there to partition a set of size n into two sets, one of which has size r ? (A *partition* of set S is a collection of subsets of S such that the intersection of any two sets in the collection is empty and the union of all the sets in the collection is the whole original set.)

Solution: $C(n, r) = \frac{n!}{r!(n-r)!}$.

5. How many ways are there to partition a set of size n into two sets A and B ? Note that it matters which set is called A and which one is called B .

Solution: One way of thinking about this problem is to use solution to the previous problem and to sum over the different sizes of A :

$$\sum_{0 \leq i \leq n} C(n, i) = \sum_{0 \leq i \leq n} \frac{n!}{i!(n-i)!}$$

However, a more direct analysis just counts the number of subsets of n , which is 2^n .

Either answer would get full points and both are correct. This reflects an important identity that we did not have time to discuss in lecture, namely

$$\sum_{0 \leq i \leq n} \frac{n!}{i!(n-i)!} = 2^n$$

6. How many 8-bit sequences have exactly 3 0's?

Solution: We just need to count the number of ways of selecting the three positions occupied by 0's:
 $C(8, 3) = \frac{8!}{3!5!}$.

7. How many ways are there to put 6 keys on a circular key ring? The two sides of the ring are the same and the two sides of the keys are the same. (So both ring and keys are symmetrical.) There is also no way to tell which key is first.

Solution: Since the two sides of the keys are the same, we don't have to consider which way they are turned. There are $6!$ ways of linearly ordering the keys. However, each arrangement of the 6 keys around the ring generates 6 different linear orderings, depending on which key you consider to be first. So, if sides of the key ring could be distinguished, there would be $\frac{6!}{6} = 5!$ ways of arranging the keys on the ring. But since each arrangement is considered the same as the (otherwise different) one obtained from it by turning over the loaded ring, we have to divide this number of 2, obtaining the answer $\frac{5!}{2}$.

8. What is the probability of being dealt a particular hand of 5 cards from a deck of 52 distinct cards?

Solution: $\frac{1}{C(52,5)}$

9. Suppose a permutation of all 26 lowercase letters is selected at random with each permutation equally likely.

- (a) What is the probability that the first 13 letters of the permutation are in order?

Solution: Whatever the first 13 letters may be, there are $13!$ ways of ordering them, only one of which is alphabetically ordered. Thus the probability is $1/13!$ A more brute force way of getting the answer is to count the number of permutations in which the first 13 letters are in order and to divide this by the total number of permutations. There are $C(26,13)$ ways of choosing 13 letters to put in the first 13 positions. For each of these choices, only one ordering of the selected letters is alphabetical. The 13 letters not chosen for the first 13 positions can be placed in $13!$ different orders. So the probability is $\frac{C(26,13)13!}{26!} = \frac{1}{13!}$

- (b) What is the probability that a and b are not next to each other?

Solution: There are $24!$ ways of ordering the letters c through z. We must choose two places to insert the letters a and b out of 25 places either between two of the other 24 letters or before or after all of them. (By choosing 2 places, we ensure a and b are not adjacent.) We must also choose which of the two places gets a and which gets b. So the number of permutations that do not have a and b adjacent is $2(24!C(25, 2))$. Dividing this by $26!$ yields $\frac{12}{13}$.