Discrete Mathematical Structures
CS 3233 Lecture Two

Prof. William Winsborough
January 17, 2008

Course Introduction

- Turn in Homework
- Questions???
- Read Sections 1.2 and 1.3 by Tuesday
- **Homework 2 due Thursday** January 24
  - Section 1.2: 2, 6, 10, 12
  - Section 1.3: 10d, 10e, 14, 24c, 24d, 32a, 32b, 44

Implications Related to p → q

- *Contrapositive*: ¬q → ¬p
- *Converse*: q → p
- *Inverse*: ¬p → ¬q

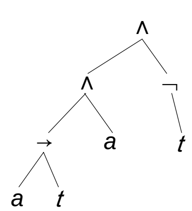
p	q	¬p	¬q	p → q	¬q → ¬p	q → p	¬p → ¬q
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Translating English to Logic

- Fred can access the wireless network only if Fred has paid his tuition
  - Let *a* represent "Fred can access wireless"
  - Let *t* represent "Fred has paid his tuition"
  - $a \rightarrow t$

Expression Trees

- Example: $((a \rightarrow t) \land a) \land \neg t$



Precedence

- Should ¬q → ¬p be interpreted as
  - (¬q) → (¬p), or as
  - ¬(q → (¬p)) ?
- Precedence gives rules for implicit parentheses

Op	Prec
¬	1
∧	2
∨	3
→	4
↔	5

A More Concise Truth Table

((a	→	t)	∧	a)	∧	¬	t
T		T		T			T
T		F		T			F
F		T		F			T
F		F		F			F

A More Concise Truth Table

((a	→	t)	∧	a)	∧	¬	t
T	T	T		T		F	T
T	F	F		T		T	F
F	T	T		F		F	T
F	T	F		F		T	F

A More Concise Truth Table

((a	→	t)	∧	a)	∧	¬	t
T	T	T	T	T		F	T
T	F	F	F	T		T	F
F	T	T	F	F		F	T
F	T	F	F	F		T	F

A More Concise Truth Table

((a	→	t)	∧	a)	∧	¬	t
T	T	T	T	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	F	F	F	F	T
F	T	F	F	F	F	T	F

Consistency

- Definition: A collection of propositional formulas is *consistent* if there is a truth assignment that makes each formula true
- Motivation from Software Engineering:
  - Software specifications should not contain conflicting requirements

Example Inconsistent Spec

- Specification:
  - Fred can access the wireless network only if Fred has paid his tuition $(a \rightarrow t)$
  - Fred can access the wireless network (a)
  - Fred has not paid his tuition $(\neg t)$

a	t	a → t	¬t	(a → t) ∧ a ∧ ¬t
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

- $(a \rightarrow t) \wedge a \wedge \neg t$ is a *contradiction*

Tautologies and Contradictions

- A compound proposition that is true for all truth assignments is a *tautology*
  - *E.g.*, $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
- One that is false for all assignments is a *contradiction*
  - *E.g.*, $p \wedge \neg p$

Logical Equivalence

- Definition: Two propositional formulas Φ and Ψ are *logically equivalent* if they have the same semantics
  - In this case we write $\Phi \equiv \Psi$
    - e.g., $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$
  - Recall that the semantics of a formula is a function from truth assignments to truth values
    - So $\Phi \equiv \Psi$ if Φ and Ψ assume the same truth value for each assignment of truth values to the propositional variables appearing in Φ and Ψ
- Theorem: Given propositional formulas Φ and Ψ, the biconditional $\Phi \leftrightarrow \Psi$ is a tautology if and only if $\Phi \equiv \Psi$
- Note that $\equiv$ is not a logical connective (*i.e.*, a logical operator), so $p \equiv q$ is not a compound proposition
- $\equiv$ is part of the *meta-language* we use for discussing formulas in the *object language* of propositional calculus

An Equivalence for Implication

Theorem:
Given any propositional formulas ϕ and ψ,
$\phi \rightarrow \psi \equiv \neg\phi \vee \psi$

Proof:

ϕ	ψ	$\phi \rightarrow \psi$	$\neg\phi \vee \psi$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Some Important Logical Equivalences

$\phi \wedge \mathbf{T} \equiv \phi$ $\phi \vee \mathbf{F} \equiv \phi$	Identity laws
$\phi \vee \mathbf{T} \equiv \mathbf{T}$ $\phi \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$\phi \vee \phi \equiv \phi$ $\phi \wedge \phi \equiv \phi$	Idempotency laws
$\neg(\neg\phi) \equiv \phi$	Double negation law
$\phi \vee \psi \equiv \psi \vee \phi$ $\phi \wedge \psi \equiv \psi \wedge \phi$	Communitivity laws

More Equivalences

$(\phi \vee \psi) \vee \theta \equiv \phi \vee (\psi \vee \theta)$ $(\phi \wedge \psi) \wedge \theta \equiv \psi \wedge (\phi \wedge \theta)$	Associativity laws
$\phi \vee (\psi \wedge \theta) \equiv (\phi \vee \psi) \wedge (\phi \vee \theta)$ $\phi \wedge (\psi \vee \theta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \theta)$	Distributivity laws
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$	De Morgan's laws
$\phi \vee (\phi \wedge \psi) \equiv \phi$ $\phi \wedge (\phi \vee \psi) \equiv \phi$	Absorption laws
$\phi \vee \neg\phi \equiv \mathbf{T}$ $\phi \wedge \neg\phi \equiv \mathbf{F}$	Negation laws

Exercise

- Use known equivalences to show the following:
  $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Section 1.3: Predicates

- Propositional functions
  - A *propositional function* p and a *predicate* p are the same thing
  - A statement of the form $p(x_1, x_2, \ldots, x_n)$ is a proposition
- Examples
  - p(x) ≡ x > 3
    - p(4) has a truth value
  - q(y) ≡ y has paid his tuition
  - r(z) ≡ z has wireless access on campus
  - What does r(x) → q(x) mean?

Predicates, Continued

- Each propositional function takes a fixed number of arguments
  - older(x,y) ≡ x is older than y
  - Here "older" is being used as a propositional function

Formulas Involving Predicates

- Let ϕ(y) ≡ q(y) → r(y)
  - We say ϕ is *a formula in* y (not in text)
- Then ϕ(Fred) ≡ q(Fred) → r(Fred)
  - This is an example of substitution of variables by values

Universe of Discourse and Quantifiers

- The *universe of discourse* or *domain* is the set of all possible values for variables
- We can refer to values in the universe either by using constant symbols (like "Fred") or by using quantifiers
- There are two quantifiers in standard predicate calculus: *for all* (∀) and *there exists* (∃)
- There are called the universal quantifier and the existential quantifier, respectively

Universal Quantifiers

- The universal quantification of p(x) is the following proposition:
  - "p(x) is true for all values of x in the universe of discourse"
  - Written ∀x p(x) or ∀x.p(x)
- Similarly, if ϕ(x) is a formula in x, ∀x.ϕ(x) means the formula holds for all elements of the universe
  - What does ∀x.(r(x) → q(x)) mean?
  - How is it different from (∀x.r(x)) → (∀x.q(x)) ?

Existential Quantifiers

- The existential quantification of p(x) is the proposition
  - "there exists an element x in the universe of discourse such that p(x) is true"
  - Written ∃x p(x) or ∃x.p(x)
  - What does ∃x.(r(x) ∧ ¬q(x)) mean?
- If the universe of discourse is {0,1,2}, then
  - ∃x.p(x) ≡ p(0) ∨ p(1) ∨ p(2)