Discrete Mathematical Structures
CS 3233 Lecture Eight

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Business

- **Recall: Homework 4**, due Thursday 2/14
  - Section 2.1: 2a, 2b, 6, 8, 18
  - Section 2.2: 2, 6, 10, 20, 26a
  - Section 2.3: 4, 10, 14, 16, 38
- Questions???

Formal Definition of Function

- A binary relation $r \subseteq A \times B$ is a *function* if
  - $\forall a \forall b_1 \forall b_2.((a,b_1) \in r \wedge (a,b_2) \in r) \rightarrow b_1 = b_2$
  - In this case, we write r(a) for b
- A function r is *total* (called a *total function*) if additionally
  - $\forall a \exists b.(a,b) \in r$

Terminology

- Let $f : A \rightarrow B$
  - We say f *maps* A to B
  - A is the *domain* of f
  - B is the *codomain* of f
  - If f(a) = b,
    - b is the *image* of a and
    - a is the *pre-image* of b
  - If $S \subseteq A$, the *image* of S is $\{f(s) \mid s \in S\}$ and is denoted by f(S)
  - $f(A) = \{f(a) \mid a \in A\}$ is called the *range* of f

Examples

- Successor function
  - $s : N \rightarrow N$
  - $s(n) = n+1$
- Floor
  - floor : $R \rightarrow Z$
  - floor(x) = greatest integer less than x
- Square root
  - $\sqrt{} : R \rightarrow C$
  - The square root of a real is a complex number
- Truth assignment
  - $\sigma : \mathcal{V} \rightarrow \{\text{true, false}\}$

Injections and Surjections

- $f : A \rightarrow B$ is *one-to-one*, or *injective*, if and only if f(x) = f(y) implies x = y for all $x, y \in A$
  - $f : A \overset{1\text{-}1}{\rightarrow} B$
  - $\forall a_1 \forall a_2 \forall b.((a_1,b) \in f \wedge (a_2,b) \in f) \rightarrow a_1 = a_2$
  - Equivalently, $\forall a_1 \forall a_2 . a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$
  - f is called an *injection*
- $f : A \rightarrow B$ is *onto,* or *surjective*, if and only if for every $b \in B$, there is an $a \in A$ such that f(a) = b
  - f is *a function of* A *onto* B
  - $f : A \overset{\text{onto}}{\rightarrow} B$
  - $\forall b \exists a.(f(a) = b)$
  - f is called a *surjection*

More Examples

- Injective and/or surjective?
  - Square
    - $f : Z \rightarrow Z$
    - $f(x) = x^2$
  - Double
    - twice : $Z \rightarrow Z$
    - twice(x) = 2x
  - Absolute value
    - $|\cdot| : Z \rightarrow N$
    - $|x| = \begin{cases} x \text{ if } 0 \leq x; \\ -x, \text{ otherwise} \end{cases}$

Still More Examples

- Injective and/or surjective?
  - Integer successor
    - zs : $Z \rightarrow Z$
    - zs(n) = n+1
  - Mapping of 32-bit words to Z
  - Successor in $Z_3 = \{0, 1, 2\}$
    - s(0) = 1
    - s(1) = 2
    - s(2) = 0

Bijections

- If $f : A \rightarrow B$ is injective and surjective, it is *bijective*
  - f is called a *one-to-one correspondence,* or a *bijection*

Compositions

- Definition
  - Given $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of f and g, $f \circ g : A \rightarrow C$, is defined by $f \circ g(a) = f(g(a))$
  - Note that no special properties of f and g are required for $f \circ g$ to be defined.
    - For instance, f and g need not be injective, surjective, or bijective
    - However, if f and g have special properties, it often follows that $f \circ g$ special properties as well
    - Study hint: think through these relationships

Function Inverses

- Theorem: Given $f : A \overset{\text{1-1}}{\underset{\text{onto}}{\rightarrow}} B$, every $b \in B$ has a unique pre-image $a \in A$
- This justifies the following definition: Given $f : A \overset{\text{1-1}}{\underset{\text{onto}}{\rightarrow}} B$
  - $f^{-1} : B \rightarrow A$ is the *inverse* of f
  - $f^{-1}(b) = a$ iff $f(a) = b$

Bijections and cardinality

- Sets A and B are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them
  - Notice that this defines a binary relation over sets
  - Contains the ordered pairs of sets between which there exists a one-to-one correspondence
- Pretty clear for finite sets