

## Discrete Mathematical Structures CS 3233 Lecture Seven

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February 7, 2008

## Business

- Turn in Homework 3
- As I indicated in my email, you do not have to turn in Homework 4 today
  - Last time we spent the whole period working exercises
- **Homework 4**, due Thursday 2/14
  - Section 2.1: 2a, 2b, 6, 8, 18
  - Section 2.2: 2, 6, 10, 20, 26a
  - Section 2.3: 4, 10, 14, 16, 38
- Questions???

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## A Final Point

- Just to really be clear about constructive versus non-constructive proofs...
  - Recall problem 8 section 1.7: prove either  $2 \cdot 10^{500} + 15$  or  $2 \cdot 10^{500} + 16$  is not a perfect square
  - Our proof in class showed
    - $\exists y. y \in \{2 \cdot 10^{500} + 15, 2 \cdot 10^{500} + 16\} \wedge \neg \exists z \in \mathbb{Z}. z^2 = y$
    - It did this by using  $p \vee \neg p$  in which  $p$  represents “ $2 \cdot 10^{500} + 15$  is a perfect square”
    - This makes it **non-constructive**
      - We do not wind up with a witness
  - If we directly were to prove that  $2 \cdot 10^{500} + 15$  is not a perfect square, this would have been a constructive proof
    - We would wind up with a witness, namely  $2 \cdot 10^{500} + 15$

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## Sets

- $s \in S$  means  $s$  is an *element* of set  $S$
- Given sets  $A$  and  $B$ ,
  - $A \subseteq B$  means  $A$  is a *subset* of  $B$ 
    - This means  $x \in A$  implies  $x \in B$
  - $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$
  - $A \subset B$  means  $A$  is a *proper subset* of  $B$ 
    - $A \neq B$
  - $A = \{a_1, a_2, \dots\}$  means that  $A$  is enumerated by the sequence  $a_1, a_2, \dots$
- *Set comprehension*: the set of values that satisfy a given property
  - Also called “set builder” notation
  - *E.g.*,  $\text{EvenInts} = \{x \mid x \in \mathbb{Z} \wedge \exists z \in \mathbb{Z}. x = 2z\}$

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## Basic Operators

- Empty set
  - $\emptyset = \{\}$
- Union
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Intersection
  - $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Set difference
  - $A - B = \{x \mid x \in A \wedge \neg x \in B\}$
- Venn diagrams
- Universe  $U$  is the set consisting of all objects under consideration
  - Resembles a universe of discourse
- Compliment
  - $\bar{A} = U - A$
- Cartesian product:
  - $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

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## Identities and Big Unions, Intersections

- Set Identities
  - Review Table 1, page 124
  - Review Example 11, page 125
- Given a collection of sets  $A_1, A_2, \dots, A_n$ 
  - $\bigcup_{1 \leq i \leq n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$
  - $\bigcap_{1 \leq i \leq n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$

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## Introduction to Functions

- Given sets  $A$  and  $B$ , a *function from  $A$  to  $B$*  is an assignment of exactly one element of  $B$  to each element of  $A$
- For  $a \in A$ , we write  $f(a) = b$  if  $b \in B$  is the unique element associated with  $a$  by  $f$
- We write  $f : A \rightarrow B$  to indicate that  $f$  is a function from  $A$  to  $B$ 
  - $A \rightarrow B$  is called the *type* of  $f$

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## Function Graphs

- This is a different usage of “graph” than either of these:
  - The plot of  $f(x)$  on the Cartesian plane
  - A set of nodes and a set of edges
- Definition
  - Given a function  $f : A \rightarrow B$ , the *graph* of  $f$  is the set of ordered pairs  $\{(a,b) \mid a \in A \text{ and } b = f(a)\}$
- Given two sets,  $A$  and  $B$ , a *binary relation*  $r$  between  $A$  and  $B$  is a subset  $r \subseteq A \times B$
- Formally speaking, a function **is** its graph
  - Thus,  $f$  is a subset of  $A \times B$  (*i.e.*, a binary relation) in which each element of  $A$  occurs exactly once

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## Formal Definition of Function

- A binary relation  $r \subseteq A \times B$  is a *function* if
  - $\forall b \forall a_1 \forall a_2. ((a_1, b) \in r \wedge (a_2, b) \in r) \rightarrow a_1 = a_2$
  - In this case, we write  $r(a)$  for  $b$
- A function  $r$  is *total* (called a *total function*) if additionally
  - $\forall a \exists b. (a, b) \in r$

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## Terminology

- Let  $f : A \rightarrow B$ 
  - We say  $f$  *maps*  $A$  to  $B$
  - $A$  is the *domain* of  $f$
  - $B$  is the *codomain* of  $f$
  - If  $f(a) = b$ ,
    - $b$  is the *image* of  $a$  and
    - $a$  is the *pre-image* of  $b$
  - If  $S \subseteq A$ , the *image* of  $S$  is  $\{f(s) \mid s \in S\}$  and is denoted by  $f(S)$
  - $f(A) = \{f(a) \mid a \in A\}$  is called the *range* of  $f$

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## Examples

- Successor function
  - $s : \mathbb{N} \rightarrow \mathbb{N}$
  - $s(n) = n+1$
- Floor
  - $\text{floor} : \mathbb{R} \rightarrow \mathbb{Z}$
  - $\text{floor}(x)$  = greatest integer less than  $x$
- Square root
  - $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{C}$
  - The square root of a real is a complex number
- Truth assignment
  - $\sigma : \mathcal{P}^2 \rightarrow \{\text{true}, \text{false}\}$

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## Injections and Surjections

- $f : A \rightarrow B$  is *one-to-one*, or *injective*, if and only if  $f(x) = f(y)$  implies  $x = y$  for all  $x, y \in A$ 
  - $f : A \xrightarrow{1:1} B$
  - $\forall a_1 \forall a_2 \forall b. ((a_1, b) \in f \wedge (a_2, b) \in f) \rightarrow a_1 = a_2$
  - Equivalently,  $\forall a_1 \forall a_2. a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$
  - $f$  is called an *injection*
- $f : A \rightarrow B$  is *onto*, or *surjective*, if and only if for every  $b \in B$ , there is an  $a \in A$  such that  $f(a) = b$ 
  - $f$  is a *function of  $A$  onto  $B$*
  - $f : A \xrightarrow{\text{onto}} B$
  - $\forall b \exists a. (f(a) = b)$
  - $f$  is called a *surjection*

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