Analysis of Algorithms

CS 3343 Lecture Seven

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Business

- Read section 4.3, 4.4.1
- Turn in: Homework 3
  - Exercise: 4.2-1
- Homework 4 Due Tuesday September 30
  - Problem 4.1 (p.85)
- I will provide a review guide Tuesday September 30
- No class Thursday Oct. 2
  - I'm giving a keynote talk at PST2008, Sixth Annual Conference on Privacy, Security and Trust, Fredericton, New Brunswick, Canada
- Tuesday October 7 will be devoted to review
  - Bring your questions
- Thursday October 9 will be midterm 1

Master Method

- Master theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$. Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

Master Example: Case 1

- $T(n) = 9T(n/3) + n$
  - $a = 9$, $b = 3$, $f(n) = n$
    $n^{\log_b a} = n^{\log_3 9} = n^2$
    $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon=1$
- So $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

Master Example: Case 2

- $T(n) = T(2n/3) + 1$
  - $a = 1$, $b = 3/2$, $f(n) = 1$
    $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$
    $f(n) = \Theta(n^{\log_b a}) = \Theta(1)$
- So $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\lg n)$

Master Example: Case 3

- $T(n) = 3T(n/4) + n \lg n$
  - $a = 3$, $b = 4$, $f(n) = n \lg n$
    $n^{\log_b a} = n^{\log_4 3} = o(n^{0.793})$
    $f(n) = \Omega(n^{\log_b a - \varepsilon})$ for $\varepsilon \approx 0.2$
  - Regularity: For sufficiently large n,
    $a f(n/b) = 3(n/4) \lg(n/4) \leq (3/4) n \lg n = c f(n)$
    for $c = 3/4$
- So $T(n) = \Theta(f(n)) = \Theta(n \lg n)$

Master Example: No Case Applies

- $T(n) = 2T(n/2) + n \lg n$
  - $a = 2$, $b = 2$, $f(n) = n \lg n$
    $n^{\log_b a} = n^{\log_2 2} = n$
    But, even though $f(n)$ is asymptotically larger than $n^{\log_b a}$, it is not polynomially larger
  - There is no constant ε such that $f(n) = \Omega(n^{\log_b a+\varepsilon})$
    Would want: $f(n) = \Omega(n^{\log_b a+\varepsilon}) \equiv f(n)/n^{\log_b a} = \Omega(n^\varepsilon)$
    But $f(n)/n^{\log_b a} = (n \lg n)/n = \lg n$ is asymptotically less than n^ε for all constants ε
- So Case 3 does not apply

Master Example: Case 2 Strong Form (From Handout)

- $T(n) = 2T(n/2) + n \lg n$
  - $a = 2$, $b = 2$, $f(n) = n \lg n$
    $n^{\log_b a} = n^{\log_2 2} = n$
    Although we do not have $f(n) = \Theta(n^{\log_b a})$, we do have
    $f(n) = \Theta(n \lg n) = \Theta(n^{\log_b a} \lg^k n)$ for $k = 1$
- So $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(n \lg^2 n)$

Reprise of My Disaster

- Problem 3.1-2: Show $(n+a)^b$ is $\Theta(n^b)$ for all real a and b where $b > 0$
  - Want constants c_1, c_2, n_0 such that for all $n \geq n_0$, $c_1 n^b \leq (n+a)^b \leq c_2 n^b$
  - Dividing by n^b, we want $c_1 \leq ((n+a)/n)^b \leq c_2$
  - Taking log base b, we want $\log_b c_1 \leq (n+a)/n \leq \log_b c_2$
  - Let $d_1 = \log_b c_1$ and $d_2 = \log_b c_2$
  - Want $d_1 \leq (n+a)/n \leq d_2$
  - For $n \geq 2|a|$, we have $½ \leq (n-|a|)/n \leq (n+a)/n \leq (n+n)/n = 2$
  - So taking $d_1 = ½$ and $d_2 = 2$, we get $c_1 = (1/2)^b$ and $c_2 = 2^b$