Discrete Mathematical Structure
Homework 4 Solution

SECTION 3.4 The Integers and Division

10. In each case we can carry out the arithmetic on a calculator.

d) Since 1 \text{ div } 23 = 0 and 1 \text{ mod } 23 = 1, we have \(-1 \text{ div } 23 = -1\) and \(-1 \text{ mod } 23 = 22\).

e) Since 2002 \text{ div } 87 = 23 and 2002 \text{ mod } 87 = 1, we have \(-2002 \text{ div } 87 = -24\) and 2002 \text{ mod } 87 = 86.

16. In each case we just apply the division algorithm (carry out the division) to obtain the quotient and remainder, as in elementary school. However, if the dividend is negative, we must make sure to make the remainder positive, which may involve a quotient 1 less than might be expected.

a) Since \(-17 = 2 \cdot (-9) + 1\), the remainder is 1. That is, \(-17 \text{ mod } 2 = 1\). Note that we do not write \(-17 = 2 \cdot (-8) - 1\), so \(-17 \text{ mod } 2 \neq -1\).

d) Since 199 = 19 \cdot 10 + 9, the remainder is 9. That is, 199 \text{ mod } 19 = 9.

SECTION 3.5 Primes and Greatest Common Divisors

4. We obtain the answers by trial division. The factorizations are 39 = 3 \cdot 13, 81 = 3^4, 101 = 101 (prime), 143 = 11 \cdot 13, 289 = 17^2, and 899 = 29 \cdot 31.

20. We form the greatest common divisors by finding the minimum exponent for each prime factor.

b) 2 \cdot 3 \cdot 11  
c) 17

SECTION 4.1 Mathematical Induction

6. The basis step is clear, since 1 \cdot 1! = 2! - 1. Assuming the inductive hypothesis, we then have

\[1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) \cdot (k + 1)! = (k + 1)! - 1 + (k + 1) \cdot (k + 1)! = (k + 1)! + (k + 1) - 1 = (k + 2)! - 1,\]
as desired.

8. The proposition to be proved is \(P(n)\):

\[2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^n = \frac{1 - (-7)^{n+1}}{4},\]

In order to prove this for all integers \(n \geq 0\), we first prove the basis step \(P(0)\) and then prove the inductive step, that \(P(k)\) implies \(P(k + 1)\). Now in \(P(0)\), the left-hand side has just one term, namely 2, and the right-hand side is \((1 - (-7)^1)/4 = 8/4 = 2\). Since 2 = 2, we have verified that \(P(0)\) is true. For the inductive step, we assume that \(P(k)\) is true (i.e., the displayed equation above), and derive from it the truth of \(P(k + 1)\), which is the equation

\[2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^k + 2 \cdot (-7)^{k+1} = \frac{1 - (-7)^{k+1}+1}{4}.\]
To prove an equation like this, it is usually best to start with the more complicated side and manipulate it until we arrive at the other side. In this case we start on the left. Note that all but the last term constitute precisely the left-hand side of $P(k)$, and therefore by the inductive hypothesis, we can replace it by the right-hand side of $P(k)$. The rest is algebra:

$$[2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^k] + 2 \cdot (-7)^{k+1} = \frac{1 - (-7)^{k+1}}{4} + 2 \cdot (-7)^{k+1}$$

$$= \frac{1 - (-7)^{k+1} + 8 \cdot (-7)^{k+1}}{4}$$

$$= \frac{1 + 7 \cdot (-7)^{k+1}}{4}$$

$$= \frac{1 - (-7) \cdot (-7)^{k+1}}{4}$$

$$= \frac{1 - (-7)^{(k+1)+1}}{4}.$$  

12. We proceed by mathematical induction. The basis step ($n = 0$) is the statement that $(-1/2)^0 = (2+1)/(3 \cdot 1)$, which is the true statement that $1 = 1$. Assume the inductive hypothesis, that

$$\sum_{j=0}^{k} \left( \frac{-1}{2} \right)^j = \frac{2^{k+1} + (-1)^{k+1}}{3 \cdot 2^k}.$$  

We want to prove that

$$\sum_{j=0}^{k+1} \left( \frac{-1}{2} \right)^j = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}.$$  

Split the summation into two parts, apply the inductive hypothesis, and do the algebra:

$$\sum_{j=0}^{k+1} \left( \frac{-1}{2} \right)^j = \sum_{j=0}^{k} \left( \frac{-1}{2} \right)^j + \left( \frac{-1}{2} \right)^{k+1}$$

$$= \frac{2^{k+1} + (-1)^{k+1}}{3 \cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}$$

$$= \frac{2^{k+2} + 2(-1)^k}{3 \cdot 2^{k+1}} + \frac{3(-1)^{k+1}}{3 \cdot 2^{k+1}}$$

$$= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}.$$  

For the last step, we used the fact that $2(-1)^k = -2(-1)^{k+1}$. 
14. We proceed by induction. Notice that the letter $k$ has been used in this problem as the dummy index of summation, so we cannot use it as the variable for the inductive step. We will use $n$ instead. For the basis step we have $1 \cdot 2^1 = (1 - 1)2^{1+1} + 2$, which is the true statement $2 = 2$. We assume the inductive hypothesis, that
\[ \sum_{k=1}^{n} k \cdot 2^k = (n - 1)2^{n+1} + 2, \]
and try to prove that
\[ \sum_{k=1}^{n+1} k \cdot 2^k = n \cdot 2^{n+2} + 2. \]
Splitting the left-hand side into its first $n$ terms followed by its last term and invoking the inductive hypothesis, we have
\[ \sum_{k=1}^{n+1} k \cdot 2^k = \left( \sum_{k=1}^{n} k \cdot 2^k \right) + (n + 1)2^{n+1} = (n - 1)2^{n+1} + 2 + (n + 1)2^{n+1} = 2n \cdot 2^{n+1} + 2 = n \cdot 2^{n+2} + 2, \]
as desired.

16. The basis step reduces to $6 = 6$. Assuming the inductive hypothesis we have
\[ 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) = \frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3) \]
\[ = (k + 1)(k + 2)(k + 3) \left( \frac{k}{4} + 1 \right) \]
\[ = (k + 1)(k + 2)(k + 3)(k + 4). \]