CS 3233 Discrete Mathematical Structure
Final Exam
Wednesday, May 5, 2004
10:30 am – 1:15 pm

Last Name: _________________
First Name: ________________
Student ID: _________________

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For problems 1-14, show every step to get partial credits.

1. (5 points) Use a truth table to show \(\neg(p \rightarrow q)\) and \(p \land \neg q\) are equivalent.

2. (10 points) Using the truth tables, prove or disprove
\[
\neg(\neg p \land q \land \neg r) \lor (p \land \neg q \land \neg r) \leftrightarrow (q \rightarrow p)
\]

3. (6 points) Compute the double summation
\[
\sum_{i=0}^{100} \sum_{j=3}^{20} (2^i + 3j)
\]
4. (5 points) Given sets $A = \{1, 2, 3, 5, 7\}$ and $B = \{4, 7, 12, 28, 52\}$ and a function $f : A \rightarrow B$ such that $f(x) = x^2 + 3$, do the following
(a) What are the domain and range of $f$? (2 points)
(b) Does $f^{-1}$ exist? If yes, give a formula for $f^{-1}$. (3 points)

5. (6 points) Determine whether the following functions are bijections from $\mathbb{R}$ to $\mathbb{R}$. Please explain.
(a) $f(x) = x^2$ (conclusion 1 point, explanation 2 points)
(b) $f(x) = x^3 - 2$ (conclusion 1 points, explanation 2 points)
6. (6 points) Let \( f : N \to N \) and \( g : N \to N \) be the functions defined by
\[
 f(x) = x^2 + x + 1 \quad \text{and} \quad g(x) = 5x + 2 ,
\]
where \( N \) is the set of natural numbers. Let \( \circ \) be the symbol for composition of functions
(a) What is \( f \circ g \) ? (3 points)

(b) What is \( g \circ f \) ? (3 points)

7. (5 points) What is the probability that a five-card poker hand contains three kings, one queen, and one 3?

8. (a) When rolling TWO dice, what is the probability of getting the sum 8? (5 points)
(b) When rolling THREE dice, what is the probability of getting the sum 7? (5 points)

9. (10 points) \( A = \{\text{Alex, Bob, Chris, Davis}\} \), \( B = \{\text{Greg, Hanna, Irene, Jim}\} \) Assume that the elements of \( A \) and \( B \) are alphabetically ordered.
   (a) Relation \( R \) is defined on \( A \times A \) and is represented by matrix
   \[
   M = \begin{pmatrix}
   1 & 0 & 1 & 1 \\
   0 & 1 & 1 & 0 \\
   1 & 0 & 0 & 1 \\
   0 & 1 & 0 & 1 \\
   \end{pmatrix}
   \]
   i. Represent \( R \) as a directed graph. (2 points)
ii. Compute the matrix $R_1^2$ and represent it as a directed graph. (4 points)

(b) Relation $R_2$ is defined on $A \times B$ and is represented as matrix $M_2$ below
Relation $R_3$ is defined on $B \times A$ and is represented as matrix $M_3$ below

\[
M_2 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} \quad M_3 = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

i. Find the matrix representing $R_2 \circ R_3$ ($\circ$ is the symbol for composite of relations). (2 points)

ii. Find the matrix representing $R_3 \circ R_2$ (2 points)
10. (5 points) Give a recursive definition of the sequences \((a_n), \ n = 1, 2, 3, \ldots\) if \(a_n = \frac{(n-1)!}{n(n+1)}\)

11. (8 points) Prove that every number greater than 17 can be written as \(4i + 7j\) where \(i\) and \(j\) are natural numbers. (hint: use the second principal of mathematical induction)
12. (8 points) Use the mathematical induction to prove that \( \sum_{i=1}^{n} \sqrt{i} > \frac{n}{2} \) for \( n \geq 1 \).

13. (10 points) Let \( R_1 \) and \( R_2 \) be relations on a set \( A \) represented by the matrices

\[
M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

Find the matrices that represent
(a) \( R_1 \circ R_2 \) (2.5 points) Note: \( \circ \) denotes the composite of relations

(b) \( R_1^2 \) (2.5 points)
14. (6 points) Find the sum-of-products (i.e., Boolean sum of distinct minterms) and product-of-sums (i.e., Boolean product of distinct maxterms) expansion for the function $F(x, y, z)$. The values of $F(x, y, z)$ are given in the following table.

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