Last Name: _________________
First Name: ________________
Student ID: _________________

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1. (10 points) Let $P, Q,$ and $R$ be propositions. Complete the truth table below for $P \rightarrow (Q \rightarrow R)$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$Q \rightarrow R$</th>
<th>$P \rightarrow (Q \rightarrow R)$</th>
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2. (10 points) Use a truth table to show that the implication $[(p \lor q) \land (p \rightarrow r) \land (q \rightarrow r)] \rightarrow r$ is a tautology

Sol:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$s = p \lor q$</th>
<th>$t = p \rightarrow r$</th>
<th>$v = q \rightarrow r$</th>
<th>$w = s \land t \land v$</th>
<th>$w \rightarrow r$</th>
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3. (14 points) Let $F(x)$ denotes “x is a freshman”, $C(x, y)$ denotes “x chooses course y” and let the universe $U$ be the set of all students. Translate the following English sentence into equivalent logical expressions.

(a) Every freshman chooses CS 1713. (3 points)

Sol: $\forall x(F(x) \rightarrow C(x, CS1713))$

(b) All students who chose CS 1713 are freshman. (3 points)

Sol: $\forall x(C(x, CS1713) \rightarrow F(x))$
(c) There is a student who did not choose CS 1713. (4 points)
**Sol:** \( \exists x (\neg C(x, \text{CS1713})) \)

(d) There is a student who chooses CS 1713 and who is not a freshman. (4 points)
**Sol:** \( \exists x (C(x, \text{CS1713}) \land \neg F(x)) \)

4. (12 points) Prove or disprove the following for sets \( A = \{1, 3, 6, 5\} \), \( B = \{3, 5, 1\} \) and \( C = \{4, 5\} \). The universe \( U = \{1, 2, 3, 4, 5, 6, 7, 8\} \)

(a) \( (A \cap B) \cup (\overline{A} \cap \overline{B}) = (\overline{A} \cup B) \cap (A \cup \overline{B}) \) (6 points)
**Sol:** This identity is true.
For the left side:
\[
A \cap B = \{1, 3, 6, 5\} \cap \{3, 5, 1\} = \{1, 3, 5\}
\]
\[
\overline{A} = \{2, 4, 7, 8\} \quad \overline{B} = \{2, 4, 6, 7, 8\} \quad \overline{A} \cap \overline{B} = \{2, 4, 7, 8\}
\]
\[
(A \cap B) \cup (\overline{A} \cap \overline{B}) = \{1, 3, 5\} \cup \{2, 4, 7, 8\} = \{1, 2, 3, 4, 5, 7, 8\}
\]
For the right side:
\[
\overline{A} \cup B = \{2, 4, 7, 8\} \cup \{3, 5, 1\} = \{1, 2, 3, 4, 5, 7, 8\}
\]
\[
A \cup \overline{B} = \{1, 3, 6, 5\} \cup \{2, 4, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}
\]
\[
(\overline{A} \cup B) \cap (A \cup \overline{B}) = \{1, 2, 3, 4, 5, 6, 8\} \cap \{1, 2, 3, 4, 5, 7, 8\} = \{1, 2, 3, 4, 5, 7, 8\}
\]

(b) \( (A - (B - C)) = ((A - B) - C) \) (6 points)
**Sol:** This identity is false.
\[
(A - (B - C)) = (\{1, 3, 6, 5\} - (\{3, 5, 1\} - \{4, 5\})) = \{1, 3, 6, 5\} - \{3, 1\} = \{6, 5\}
\]
\[
((A - B) - C) = ((\{1, 3, 6, 5\} - \{3, 5, 1\}) - \{4, 5\}) = (\{6\} - \{4, 5\}) = \{6\}
\]

5. (10 points) Use the Venn Diagram to prove \( (B - A) \cup (C - A) = (B \cup C) - A \)
**Sol:**

\[
\text{Venn Diagram for (B - A) \cup (C - A)}
\]
\[
\text{Venn Diagram for (B \cup C) - A}
\]
6. (10 points) Let \( f : A \rightarrow B \) be a function with \( A = \{\text{cow, automobile, 3}\} \), \( B = \{\text{banana, } \pi, \text{Elvis, sky}\} \).

\( F \) is defined by:

\[
\begin{align*}
\ f(\text{cow}) &= \text{Elvis} \\
\ f(\text{automobile}) &= \pi \\
\ f(3) &= \text{banana}
\end{align*}
\]

(a) What is the range and domain of \( f \)? (5 points)

**Sol:** The range of \( f \) is \( \{\text{Elvis, } \pi, \text{banana}\} \).
The domain of \( f \) is \( \{\text{cow, automobile, 3}\} \).

(b) Is \( f \) a bijection? Justify your answer. (5 points)

**Sol:** No, it is not a onto function because no element of the domain is associated with sky.

7. (12 points) Let \( f : N \rightarrow N \) and \( g : N \rightarrow N \) be the functions defined by

\[ f(x) = 3x + 4 \] and \( g(x) = 2x + 1 \), where \( N \) is the set of natural numbers. Let \( \circ \) be the symbol for composition of functions

(a) What is \( f \circ g \)? (6 points)

**Sol:**

\[
\begin{align*}
\ f \circ g (x) &= f (g(x)) = f (2x + 1) = 3*(2x + 1) + 4 = 6x + 7
\end{align*}
\]

(b) What is \( g \circ f \)? (6 points)

**Sol:**

\[
\begin{align*}
\ g \circ f (x) &= g (f(x)) = g (3x + 4) = 2*(3x + 4) + 1 = 6x + 9
\end{align*}
\]
8. (10 points) Let \( f(n) = (n \log n + n)(n^2 + 1) \), \( g(n) = n^3 \log n \), prove that \( f(n) = O(g(n)) \) by specifying \( C \) and \( K \).

**Sol:**

\[
f(n) = (n \log n + n)(n^2 + 1) = n^3 \log n + n^6 + n \log n + n
\]

\[g(n) = n^3 \log n\]

To prove \( f(n) \) is \( O(g(n)) \)

**Step 1:** Choose \( k = 10 \)

**Step 2:** Assume \( n > 1 \)

Find/derive such a \( C \) such that the ratio of \( \frac{f(n)}{g(n)} \leq C \)

\[
\frac{f(n)}{g(n)} = \frac{n^3 \log n}{n^3 \log n} + \frac{n^3}{n^3 \log n} + \frac{n \log n}{n^3 \log n} + \frac{n}{n^3 \log n} \leq C
\]

\[
1 + \frac{1}{\log n} + \frac{1}{n^2} + \frac{1}{n^2 \log n} \leq C
\]

Each term on the left is less than or equal to one for every \( n > 10 \) so the sum must be less than four. Therefore, \( C = 4 \) is sufficient.

The answer is: \( f(n) = O(g(n)) \) with \( k = 10 \) and \( C = 4 \).

9. (12 points) Compute the sum: \( \sum_{i=1}^{7} \sum_{j=3}^{10} (2 \cdot i + 3 \cdot 2^j + i \cdot 3^j) \)

**Sol:**

\[
\sum_{i=1}^{7} \sum_{j=3}^{10} (2 \cdot i + 3 \cdot 2^j + i \cdot 3^j) = \sum_{i=1}^{7} 2 \cdot i \cdot 10 + \sum_{i=1}^{7} 3 \cdot 2^j + \sum_{i=1}^{7} i \cdot 3^j
\]

\[
= 2 \cdot \frac{7^{(7+1)}}{2} \cdot 8 + 7 \cdot 3 \left( \sum_{j=0}^{10} 2^j - \sum_{j=0}^{2} 2^j \right) + \frac{7^{(7+1)}}{2} \cdot \left( \sum_{j=0}^{10} 3^j - \sum_{j=0}^{3} 3^j \right)
\]

\[
= 7 \cdot 8 \cdot 8 + 21 \left( \frac{2^{11} - 1}{2 - 1} - \frac{2^{3} - 1}{2 - 1} \right) + 28 \left( \frac{3^{11} - 1}{3 - 1} - \frac{3^{3} - 1}{3 - 1} \right)
\]

\[
= 448 + 21 \cdot (2^{11} - 2^3) + 14 \cdot (3^{11} - 3^3)
\]

\[
= 2522968
\]