For problems 1-6, show details to have partial credits.

1. (15 points) Let \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \) \( B = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \), find
   (a) the inverse \( A^{-1} \) so that \( A \cdot A^{-1} = I \) (5 points)
   \[
   A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ if } ac - bd \neq 0 \text{ then } \\
   A^{-1} = \begin{bmatrix} \frac{d}{ac-bd} & \frac{-b}{ac-bd} \\ \frac{-c}{ac-bd} & \frac{a}{ac-bd} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}
   \]
   (b) \( A^2 \) (4 points)
   \[
   A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}
   \]
   (c) \( A \cdot B \) and \( B \cdot A \) (6 points)
   \[
   A \cdot B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}
   \]
   \[
   B \cdot A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}
   \]

2. (15 points)
   (a) Find the prime factorization of 105 (5 points)
   \[
   \text{Sol: } 7 \cdot 3 \cdot 5
   \]
   (b) Find the greatest common divisor and least common multiple of the pair integers:
      \[2 \cdot 3^2 \cdot 5 \cdot 11 \text{ and } 3^2 \cdot 5^3 \cdot 7 \cdot 13^2 \] (5 points)
   \[
   \text{gcd}(2 \cdot 3^2 \cdot 5 \cdot 11, 3^2 \cdot 5^3 \cdot 7 \cdot 13^2) = 2^{\min(1,0)} \cdot 3^{\min(2,2)} \cdot 5^{\min(1,3)} \cdot 7^{\min(0,1)} \cdot 11^{\min(1,0)} \cdot 13^{\min(0,2)} \\
   = 2^0 \cdot 3^2 \cdot 5^1 \cdot 7^0 \cdot 11^0 \cdot 13^0 = 45
   \]
   \[
   \text{lcm}(2 \cdot 3^2 \cdot 5 \cdot 11, 3^2 \cdot 5^3 \cdot 7 \cdot 13^2) = 2^{\max(1,0)} \cdot 3^{\max(2,2)} \cdot 5^{\max(1,3)} \cdot 7^{\max(0,1)} \cdot 11^{\max(1,0)} \cdot 13^{\max(0,2)} \\
   = 2^1 \cdot 3^2 \cdot 5^3 \cdot 7^1 \cdot 11^1 \cdot 13^1 = 2252250
   \]
   (c) Convert an octal number \((176)_8\) to its decimal notation. (5 points)
   \[
   \text{Sol: } (176)_8 = 1 \cdot 8^2 + 7 \cdot 8^1 + 6 \cdot 8^0 = 64 + 56 + 6 = 126
   \]
3. (20 points) Prove that the square of an even number is an even number:
   (a) Using a directed proof (hint: proof of $p \rightarrow q$) (10 points)
   **Sol:** Let $p: n$ be an even number, and $q: n^2$ is even
   If $n$ is even, then $n=2k$.
   Therefore, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is even

   (b) Using an indirect proof (hint: an indirect proof for a proposition $p \rightarrow q$ is a direct proof of its contrapositive $\neg q \rightarrow \neg p$) (10 points)
   **Sol:** $\neg q : n^2$ is odd number, $\neg p : n$ is odd number

   If $n^2$ is odd number, then $n^2=2k+1$, therefore $n$ is odd;
   Otherwise, if $n$ is even, $n^2$ is even.

4. (15 points) What is the probability that a five-card poker hand contains two kings, two queens, and one card between 2 and 5 (including 2 and 5)?

   **Sol:** There are $C(4, 2)$ ways of choosing the kings.
   There are $C(4, 2)$ ways of choosing the queens.
   There are $C(16, 1)$ ways of choosing a card between 2 and 5.
   There are total of $C(4, 2)C(4, 2)C(16, 1) = 6 \times 6 \times 16 = 576$ possible hands with two kings, two queens, and one card between 2 and 5. This is the event space.

   The sample space is $C(52, 5)=2598960$

   The probability is $\frac{C(4,2)C(4,2)C(16,1)}{C(52,5)} = \frac{576}{2598960} = 0.0002216$
5. (20 points) Use mathematical induction to prove 2 divides \((n^2 - 1)(n + 2)\) for any positive integer \(n\).

**Sol:** Let the statement \(P(n)\) be \(2 \mid (n^2 - 1)(n + 2)\)
- **Basis step:** Prove \(P(1)\) is true.
  
  \[
  (1^2 - 1)(1 + 2) = 0 \cdot 3 = 0 \quad \text{which is divided by 2, so } P(1) \text{ is true.}
  \]

- **Inductive step:** We want to prove \(P(n) \rightarrow P(n + 1)\), which is
  
  If \(2 \mid (n^2 - 1)(n + 2)\), then \(2 \mid ((n + 1)^2 - 1)((n + 1) + 2)\)
  
  \[
  ((n + 1)^2 - 1)((n + 1) + 2) = (n^2 + 2n + 1 - 1)(n + 3)
  = n(n + 2)(n + 3) = (n^2 + 3n)(n + 2) = (n^2 - 1 + 3n + 1)(n + 2)
  = (n^2 - 1)(n + 2) + (3n + 1)(n + 2)
  
  According to \(P(n)\), the first term is divisible by 2. For the second term, one of factors must be even, so is the product. As the result, \(P(n + 1)\) is true.

6. (15 points) Give a recursive definition of the sequence \(\{a_n\}, n = 2, 3, 4, \ldots \) if

\[
a_n = \frac{n!}{(n-1)!}
\]

**Sol:**

1. **starting point:** \(a_2 = 2\)

2. \(a_n = \frac{n!}{(n-1)!} \quad \text{and} \quad a_{n-1} = \frac{(n-1)!}{(n-2)!}\)

   Consider the ratio \(\frac{a_n}{a_{n-1}} = \frac{\frac{n!}{(n-1)!}}{\frac{(n-1)!}{(n-2)!}} = \frac{n(n-2)}{(n-1)!} \), so \(a_n = \frac{n(n-2)}{(n-1)!} a_{n-1}\)