CS 3233 Discrete Mathematical Structure
Midterm 2 Exam
Tuesday, April 5, 2005
12:30 – 1:45 pm

Last Name: _________________
First Name: ________________
Student ID: _________________

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For problems 1-8, show details to have partial credits.

1. (10 points) Find $AB$ and $B^T \cdot A$ where $T$ means transpose of matrix if

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
1 & 2 & 2 \\
2 & 1 & -1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & -1 & 2 & 3 \\
-1 & 0 & 3 & -1 \\
-3 & -2 & 0 & 2
\end{bmatrix}
\]

Sol.

\[
AB = \begin{bmatrix}
4 & -1 & -7 & 6 \\
-7 & -5 & 8 & 5 \\
4 & 0 & 7 & 3
\end{bmatrix}
\]

\[
B^T \cdot A = \begin{bmatrix}
-6 & -8 & 1 \\
-5 & 1 & 2 \\
5 & 0 & 6 \\
6 & -9 & -4
\end{bmatrix}
\]

2. (15 points)
   (a) What are the quotient and remainder when (i) -123 is divided by 19 (ii) 777 is divided by 21 (5 points)

   Sol: -123 = (-7) (19) + 10, the quotient is -7 and remainder is 10.
   
   \[777 = (37) (21), \text{the quotient is 37 and the remainder 777} \mod 21 = 0.\]

   (b) Find the greatest common divisor and least common multiple of the pair integers: $11 \cdot 13 \cdot 17$ and $2^3 \cdot 3 \cdot 11 \cdot 17^2$ (5 points)

   Sol: $\gcd(11 \cdot 13 \cdot 17, \ 2^3 \cdot 3 \cdot 11 \cdot 17^2) = 2^{\min(0,3)} \cdot 3^{\min(0,1)} \cdot 11^{\min(1,1)} \cdot 13^{\min(1,0)} \cdot 17^{\min(1,2)}$
   \[= 2^0 \cdot 3^0 \cdot 11^1 \cdot 13^0 \cdot 17^1 = 11 \cdot 17 = 187\]

   $\text{lcm}(11 \cdot 13 \cdot 17, \ 2^3 \cdot 3 \cdot 11 \cdot 17^2) = 2^{\max(0,3)} \cdot 3^{\max(0,1)} \cdot 11^{\max(1,1)} \cdot 13^{\max(1,0)} \cdot 17^{\max(1,2)}$
   \[= 2^3 \cdot 3^1 \cdot 11^1 \cdot 13^1 \cdot 17^2 = 991848\]

   (c) Convert a hexadecimal number $(2AE0B)_{16}$ to its decimal notation. (5 points)

   Sol: $(2AE0B)_{16} = 2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16 + 11 = (175627)_{10}$
3. (15 points) Let \( n \) be an odd number. Prove that \( n^3 + 2n^2 \) is also odd.

(a) Using a directed proof (hint: proof of \( p \rightarrow q \)) (8 points)

**Sol:** \( n \) is an odd number, i.e., \( n = 2k + 1 \), where \( k \) is an integer.
\[
\begin{align*}
n^3 + 2n^2 &= (2k + 1)^3 + 2(2k + 1)^2 \\
&= 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1
\end{align*}
\]

Let \( k' = 4k^3 + 6k^2 + 3k \), then \( n^3 + 2n^2 = 2k'+1 \), where \( k' \) is an integer.
Therefore, \( n^3 + 2n^2 \) is an odd number.

(b) Using an indirect proof (hint: an indirect proof for a proposition \( p \rightarrow q \) is a direct proof of its contrapositive \( \neg q \rightarrow \neg p \)) (7 points)

**Sol:** Assume \( n^3 + 2n^2 \) is even, we need to prove the \( n \) is also even.
Then \( n^3 + 2n^2 = 2k \), where \( k \) is an integer. We have \( n^3 = 2k - 2n^2 \)
Since \( 2n^2 \) is an even number, so \( n^3 \) has to be even number, so is \( n \).

4. (15 points) Use mathematical induction to prove that \[ \sum_{i=1}^{n} \sqrt{i} > \frac{n\sqrt{n}}{2} \] for \( n \geq 1 \).

**Sol:** Let the statement \( P(n) \) means \( \sum_{i=1}^{n} \sqrt{i} > \frac{n\sqrt{n}}{2} \)

**Basis step:** \( P(1) \) is true because \( \sum_{i=1}^{1} \sqrt{i} = 1 > \frac{1\sqrt{1}}{2} = \frac{1}{2} \)

**Inductive step:** We want to prove that \( P(n) \rightarrow P(n+1) \), which is

If \( \sum_{i=1}^{n} \sqrt{i} > \frac{n\sqrt{n}}{2} \) is true, then \( \sum_{i=1}^{n+1} \sqrt{i} > \frac{(n+1)\sqrt{n+1}}{2} \) is true.

Using backward reasoning,
\[
\begin{align*}
\sum_{i=1}^{n+1} \sqrt{i} &> \frac{(n+1)\sqrt{n+1}}{2} \\
\sum_{i=1}^{n} \sqrt{i} + \sqrt{n+1} &> \frac{(n+1)\sqrt{n+1}}{2} \\
\sum_{i=1}^{n} \sqrt{i} &> \frac{(n+1)\sqrt{n+1}}{2} - \sqrt{n+1} \\
\frac{n\sqrt{n}}{2} &> \frac{(n+1)\sqrt{n+1}}{2} - \sqrt{n+1} \\
n\sqrt{n} &> (n+1)\sqrt{n+1} - 2\sqrt{n+1} \\
n\sqrt{n} &> (n-1)\sqrt{n+1} \\
\sqrt{n^3} &> \sqrt{(n-1)(n-1)(n+1)} \\
\sqrt{n^3} &> \sqrt{(n-1)(n^2 - 1)}
\end{align*}
\]
5. (15 points) Show by mathematical induction that $n^5 - n$ is divisible by 5 whenever $n$ is a nonnegative integer.

**Sol:** **Basis step:** We show the base case of $n = 0$.

$$n^5 - n = 0^5 - 0 = 0$$

which is divisible by 5. It is a mistake to use a base case of $n = 1$ or $n = 2$ because we must prove the proposition for all *nonnegative* integers, thus we must begin at the lowest such number, zero.

**Inductive step:** let’s assume that $5 \mid n^5 - n$, we want to prove that

$$5 \mid (n + 1)^5 - (n + 1).$$

$$(n + 1)^5 - (n + 1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - (n + 1)$$

Let us reorganize the terms a bit:

$$(n + 1)^5 - (n + 1) = n^5 - n + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - 1$$

$$= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n)$$

We are left with one term, assumed to be divisible by 5 from our inductive hypothesis, and another term which is also clearly divisible by 5. Trivially, the sum of two such terms must also be divisible by 5, therefore our hypothesis stands and we accept the proposition.

6. (5 points) Give a recursive definition of the sequences $(a_n)$, $n = 1, 2, 3, ...$ if $a_n = \frac{(n-1)!}{n(n+1)}$

**Sol:** There are multiple answers.

(i) starting point $a_1 = 1/2$

(ii) $a_n = \frac{(n-1)!}{n(n+1)}$ and $a_{n-1} = \frac{(n-2)!}{(n-1)n}$

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)!}{n(n+1)} \cdot \frac{n(n-1)}{(n-2)!} = \frac{(n-1)!}{(n-2)!} \cdot \frac{n(n-1)}{n+1} = \frac{(n-2)!}{(n-1)n} \cdot \frac{n(n-1)}{n+1}$$

Therefore $a_n = \frac{(n-1)^2}{n+1} a_{n-1}$

7. (10 points) Elections

(a) A school district has an opening on the school board. There are six candidates. In how many different orderings can the candidates’ name be listed on the ballot? (5 points)

**Sol:** This is a simple permutation problem. There are $P(6,6) = 6! = 720$ ways to list the name on the ballot.

(b) Suppose instead that the school district has three wards. There are three candidates for ward 1, four candidates for ward 2, and two candidates for ward 3.
If the wards are listed in the natural order on the ballot, in how many different orderings can the candidates’ names be listed on the ballot? (5 points)

**Sol:** There are $P(3,3) = 3! = 6$ ways to list the candidates for ward 1, $P(4,4) = 4! = 24$ ways to list the candidates for ward 2, and $P(2,2) = 2! = 2$ ways to list the candidates for ward 3. Since these ward listings are independent, there are $6 \cdot 24 \cdot 2 = 288$ ways to create the ballot.

8. (15 points) What is the probability of being dealt a 4-card hand from a standard 52-card deck having two cards of the same kind (i.e., two sevens, or two queens, etc.) and the remaining two cards being different kinds (from each other and from the matching pair)?

**Sol:** There are $C(13,1)$ ways to choose the kind of card to repeat, and $C(4,2)$ ways to choose the two cards of the same kind. There are $C(12,2)$ ways to choose which two kinds of cards to finish the hand. For each of these hands, there are $C(4,1)$ ways to pick one of the four cards of that kind. The probability of the hand is therefore

$$P(E) = \frac{|E|}{|S|} = \frac{C(13,1) \cdot C(4,2) \cdot C(12,2) \cdot C(4,1) \cdot C(4,1)}{C(52,4)}$$

$$= \frac{13 \cdot \frac{4!}{2!2!} \cdot \frac{12!}{2!10!} \cdot 4 \cdot 4 \cdot 13 \cdot 6 \cdot 66 \cdot 4 \cdot 4}{13 \cdot 17 \cdot 25 \cdot 49} \approx \frac{6336}{20825} \approx 0.304$$