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This is a closed book, closed notes exam with one exception: you may use a sheet (both sides) of your notes and summary, and the use of calculator is allowed.

Answer question worth 70 points or more (total score is 80). The exam will be graded for a maximum score of 70 points. Show all the major steps in your work.

1. (10 points) How many strings of four decimal digits
   a) do not contain the same digit twice? (4 points)
   b) end with an even digit? (3 points)
   c) have exactly three digits that are 9s? (3 points)

   **Sol:** (a) There are 10 ways to choose the first digit, 9 ways to choose the second, and so on; therefore the answer is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$;

   (b) There are 10 ways to choose the first three digits and 5 ways to choose the last; therefore the answer is $10 \cdot 10 \cdot 10 \cdot 5 = 5000$

   (c) There are 4 ways to choose the position that is to be different from 9, and 9 ways to choose the digit to go there. Therefore there are $4 \cdot 9 = 36$ such strings.

2. (10 points)
   (a) What are the quotient and remainder when (i) -109 is divided by 13 (ii) 451 is divided by 17 (5 points)

   **Sol:** $-109 = -9 \cdot 13 + 8, \ q = -9$ and $r = 8$

   $451 = 26 \cdot 17 + 9, \ q = 26$ and $r = 9$

   (b) Find the greatest common divisor and least common multiple of the pair integers:
   $3 \cdot 5^3 \cdot 7 \cdot 11$ and $2^3 \cdot 3^3 \cdot 11^2 \cdot 17$ (5 points)

   **Sol:**

   $gcd = 2^{\min(0,3)} \cdot 3^{\min(1,3)} \cdot 5^{\min(2,0)} \cdot 7^{\min(1,0)} \cdot 11^{\min(1,2)} \cdot 17^{\min(0,1)} = 3 \cdot 11 = 33$

   $lcm = 2^{\max(0,3)} \cdot 3^{\max(1,3)} \cdot 5^{\max(2,0)} \cdot 7^{\max(1,0)} \cdot 11^{\max(1,2)} \cdot 17^{\max(0,1)} = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 17 = 77754600$
3. (10 points) Find \( f(1), f(2), f(3), f(4), \) and \( f(5) \) if \( f(n) \) is defined recursively by \( f(0) = 3 \) and for \( n = 0, 1, 2, \ldots \)
   a) \( f(n + 1) = f(n)^2 - 2f(n) - 2 \) (5 points)
   b) \( f(n + 1) = 3^{f(n)/3} \) (5 points)

**Sol:** (a)

\[
\begin{align*}
  f(1) &= f(0)^2 - 2f(0) - 2 = 3^2 - 2 \cdot 3 - 2 = 1 \\
  f(2) &= f(1)^2 - 2f(1) - 2 = 1^2 - 2 - 2 = -3 \\
  f(3) &= f(2)^2 - 2f(2) - 2 = (-3)^2 - 2(-3) - 2 = 13 \\
  f(4) &= f(3)^2 - 2f(3) - 2 = 13^2 - 2 \cdot 13 - 2 = 141 \\
  f(5) &= f(4)^2 - 2f(4) - 2 = 141^2 - 2 \cdot 141 - 2 = 19,597
\end{align*}
\]

(b) First note \( f(1) = 3^{f(0)/3} = 3^{3/3} = 3 = f(0) \). In the same manner, \( f(n) = 3 \) for all \( n \).

4. (15 points) Use mathematical induction to prove that

\( 1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3 \) whenever \( n \) is a nonnegative integer.

**Sol:** (i) **Basis step.** Prove \( P(0) \) true

\[
LHS = 1; \quad RHS = (0 + 1)(2 \cdot 0 + 1)(2 \cdot 0 + 3)/3 = 3/3 = 1
\]

Therefore \( P(1) \) is true.

(ii) **Inductive Step.** Assume \( P(n) \) true. Consider \( P(n + 1) \)

\[
1^2 + 3^2 + \cdots + (2n + 1)^2 + (2(n + 1) + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3 + (2(n + 1) + 1)^2
\]

\[
= (n + 1)(2n + 1)(2n + 3)/3 + (2n + 3)^2 = \frac{(2n+3)(n+1)(2n+1)+3(2n+3)}{3} = \frac{(2n+3)(2n^2+3n+9+6n+9)}{3}
\]

Therefore \( P(n + 1) \) is true.

Hence, for all \( n \), \( P(n) \) is true.
5. (10 points) What is the probability of these events when we randomly select a permutation of \( \{1, 2, \cdots, n\} \) where \( n \geq 4 \)?
   a) 1 precedes 2. (2.5 points)
   b) 1 immediately precedes 2. (2.5 points)
   c) \( n \) precedes 1 and \( n-1 \) precedes 2. (2.5 points)
   d) \( n \) precedes 1 and \( n \) precedes 2. (2.5 points)

**Sol:** We exploit symmetry in answering many of these.

(a) Since 1 has either to precede 2 or to follow it, and there is no reason that one of these should be any more likely than the other, we immediately see the answer is \( \frac{1}{2} \).

(b) For 1 immediately to precede 2, we can think of these two numbers as glued together in forming the permutation. Then we are really permuting \( n-1 \) numbers – the single numbers from 3 through \( n \) and the one glued object. There are \( (n-1)! \) ways to do this. Since there are \( n! \) permutations in all, the probability of randomly selecting one of these is \( \frac{(n-1)!}{n!} = \frac{1}{n} \).

(c) Half of the permutations have \( n \) preceding 1. Of these permutations, half of them have \( n-1 \) preceding 2. Therefore one fourth of the permutations satisfy these conditions, so the probability is \( \frac{1}{4} \).

(d) Looking at the relative placements of 1, 2, and \( n \), we see that one third of the time, \( n \) will come first. Therefore the answer is \( \frac{1}{3} \).

6. (10 points) Assume the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has:
   a) exactly three boys? (2 points)
   b) at least one boy (3 points)
   c) at least one girl (2 points)
   d) all children of the same sex? (3 points)

**Sol:** These questions are applications of binomial distributions. We call having a boy success. Then \( p = 0.51 \) and \( n = 5 \) for this problem.

(a) The answer is: \( C(5,3) \cdot p^3 \cdot (1-p)^2 = C(5,3) \cdot 0.51^3 \cdot 0.49^2 \approx 0.32 \)

(b) There will be at least one boy if there are not all girls. The probability of all girls is \( 0.49^5 \), so the answer is \( 1 - 0.49^5 \approx 0.972 \).

(c) This is just like part (b). The answer is \( 1 - 0.51^5 \approx 0.965 \).

(d) There are two ways this can happen. The answer is clearly \( 0.51^5 + 0.49^5 \approx 0.063 \).
7. (15 points) What is the probability of that a 5-card poker hand contains (the total number of cards is 52)
   (a) all 5 hearts (2 points)
   (b) 5 cards of the same suit? (2 points)
   (c) three spades and two clubs? (3 points)
   (d) two cards of one suit and three cards of a second suit. (2 points)
   (e) one diamond, two hearts, one spade and one club? (3 points)
   (f) two cards of one suit, one card of a second suit, one card of a third suit, and one card of the fourth suit? (3 points)

Sol: (a) \( \text{Prob} = \frac{\binom{13}{5}}{\binom{52}{5}} \)
(b) \( \text{Prob} = \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} \)
(c) \( \text{Prob} = \frac{\binom{13}{3} \cdot \binom{13}{2}}{\binom{52}{5}} \)
(d) \( \text{Prob} = \frac{\binom{4}{1} \cdot \binom{13}{2} \cdot \binom{3}{1} \cdot \binom{13}{1}}{\binom{52}{5}} \)
(e) \( \text{Prob} = \frac{\binom{13}{1} \cdot \binom{13}{2} \cdot \binom{13}{1} \cdot \binom{13}{1}}{\binom{52}{5}} \)
(f) \( \text{Prob} = \frac{\binom{4}{1} \cdot \binom{13}{2} \cdot \binom{3}{1} \cdot \binom{13}{1} \cdot \binom{2}{1} \cdot \binom{13}{1} \cdot \binom{13}{1}}{\binom{52}{5}} \)