Matrices 2/1/12

Note Title

A matrix is a rectangular array of numbers organized in rows and columns.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$ is a matrix.

row

column

3 rows x 3 columns

$\therefore$ A is a 3x3 matrix

$$A = \begin{pmatrix} 0 & -1 & 5 & 2 \\ 7 & 2 & -20 & 100 \end{pmatrix}$$ is a 2x4 matrix.

or A is of order 2x4.

In general, a matrix A is of the order $m \times n$ means

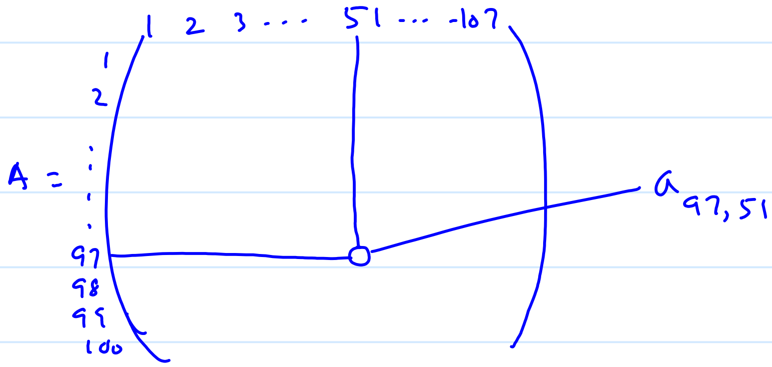
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$a_{i1}\ a_{i2} \cdots a_{in} \leftarrow$ row i

$$\left.\begin{matrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{matrix}\right\}$$ is column j

$a_{i,j}$ - an element of A in row i and col. j

Suppose A is an 100x107 matrix. Where is $a_{97,51}$?



$A = (a_{ij})_{m \times n}$

denotes matrix A of order $m \times n$.

element i, j is given by $a_{i,j}$.

Special matrices

If a matrix has only one row, then it is a row vector.

$A = \begin{pmatrix} 1 & 10 & 11 & 12 & 7 \end{pmatrix}$ is a 1×5 matrix.

It is a row matrix or row vector.

If a matrix has only one column, then it is a column vector. $B = \begin{pmatrix} 7 \\ -2 \\ 5 \\ 11 \end{pmatrix}$ is a 4x1 matrix or a column vector.

A 1x1 matrix is a scalar. $C = \begin{pmatrix} 7 \end{pmatrix}_{1 \times 1}$ is a 1x1 matrix or a scalar.

A null matrix has 0 for all of its entries. $A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$ is a null matrix.

If the number of rows of a matrix is the same as the number of its columns, then it is a square matrix.

$A = (a_{ij})_{n \times n}$ #rows = #columns $\Rightarrow$ A is a square matrix.

The main diagonal of a matrix consists of the elements whose row and column indices are the same. Therefore, the main diagonal starts with the top left corner element, and goes through the elements in the southeast direction.

Main diagonal is defined for square and nonsquare matrices. However, it is more interesting for square matrices.

If $A = (a_{ij})_{n \times n}$, then elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the main diagonal of A.

main diagonal.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -6 & 7 & 2 \\ 2 & -5 & 10 \end{pmatrix}$$

An **identity matrix** is a square matrix that has 1s on the main diagonal and 0s everywhere else. An identity matrix of order nxn is denoted by I_n.

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix}_{n \times n} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A **diagonal matrix** is a square matrix that may have nonzero entries only on the main diagonal. (That is 0s in all off-main-diagonal positions.) A diagonal matrix is denoted by D_n.

$D_n = (d_{ij})_{nxn}$ where $d_{ij} = 0$ if $i \neq j$

$$D_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \qquad D_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

A **lower triangular matrix** is a square matrix that may nonzero entries only on the main diagonal and below the main diagonal. A lower triangular matrix of order nxn is denoted by L_n.

An **upper triangular matrix** is a square matrix that may have nonzero entries only on the main diagonal and above the main diagonal. An upper triangular matrix of order nxn is denoted by U_n.

LT

$$\begin{pmatrix} 1 & & & & \\ 0 & 2 & & 0 & \\ 1 & -1 & 3 & & \\ 0 & 0 & 2 & 4 & \\ 0 & 1 & 2 & 3 & 5 \end{pmatrix}$$

L_5

UT

$$\begin{pmatrix} 2 & 0 & 1 & 2 & 3 \\ & -7 & 0 & 1 & 1 \\ & & 5 & 0 & 2 \\ & 0 & & 0 & 5 \\ & & & & 1 \end{pmatrix}$$

U_5

$L_n = (l_{ij})_{nxn}$ where $l_{ij} = 0$ if $i > j$.

$U_n = (u_{ij})_{nxn}$ where $u_{ij} = 0$ if $i < j$.

A **Boolean** or **binary matrix** has only 1s and 0s as its entries.

has only 0s and 1s as its entries.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

A row or right **stochastic matrix** is a square matrix with nonnegative entries ≤ 1; and the sum of all the entries in each row is exactly 1.

$$\begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.7 & 0.3 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

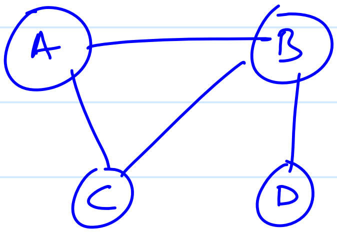
all entries are in $[0,1]$ and.

each row sums to 1.

$\Rightarrow$ row stochastic matrix.

Applications

1. Adjacency matrix of a graph.



Adjacency matrix of the graph

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Adjacency matrices can be manipulated to deduce some graph properties and to achieve some graph operations.

2. System of linear equations

$$\left.\begin{aligned} 3x_1 + 4x_2 &= 7 \\ -2x_1 + 7x_3 &= 9 \\ 2x_1 + 3x_2 + 5x_3 &= 20 \end{aligned}\right\}$$

Can be represented using matrices & vectors.

$$\begin{array}{r} \text{Eq. 1} \rightarrow \\ \text{Eq. 2} \rightarrow \\ \text{Eq. 1} \rightarrow \end{array} \begin{pmatrix} 3 & 4 & 0 \\ -2 & 0 & 7 \\ 2 & 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 20 \end{pmatrix}$$

Coefficients of variables   Vector of variables   Vector of constants on the RHS

Equal matrices

Let $A=(a_{i,j})_{mxn}$ and $B=(b_{i,j})_{pxq}$ be two matrices.

A = B if and only if

(i) A and B are of the same order; that is, m=p and n=q

(ii) $a_{i,j} = b_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times3} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}_{2\times3}$$

$A = B$ iff $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{13} = b_{13}$

$a_{21} = b_{21}$ $a_{22} = b_{22}$ $a_{23} = b_{23}$

Matrix Addition

Addition of two matrices A and B, denoted A+B, is defined if A and B are of the same order.

A+B is obtained by adding the same position elements of A and B.

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{pmatrix}$$

is A+B defined? Yes

$$A + B = \begin{pmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{pmatrix}$$

If A and B are of order $m \times n$,

$A = (a_{ij})_{m \times n}$ $\quad B = (b_{ij})_{m \times n}$

$A + B = \left(a_{ij} + b_{ij}\right)_{m \times n}$ $\quad A - B = \left(a_{ij} - b_{ij}\right)_{m \times n}$

$$\begin{pmatrix} 8 & 3 & 6 \\ 5 & 2 & 7 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 8-1 & 3-2 & 6-3 \\ 5-4 & 2-5 & 7-6 \\ 1-7 & 0-8 & 4-9 \end{pmatrix}$$

Scalar-Matrix Product

$$4 \cdot \begin{pmatrix} 3 & 2 & 5 \\ 6 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 4\times3 & 4\times2 & 4\times5 \\ 4\times6 & 4\times1 & 4\times7 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 20 \\ 24 & 4 & 28 \end{pmatrix}$$

In general $k \cdot (a_{ij})_{m \times n} = \left(k \cdot a_{ij}\right)_{m \times n}$

$$2 \cdot I_3 = 2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Dot product

A dot product is a multiplication of a row vector of order 1xn with a column vector of order nx1. The result is a scalar. It is obtained by multiplying ith element of the row vector with ith element of the column vector and summing these products.

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$A \cdot B = a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n$$

Examples.

$$A = \begin{pmatrix} 1 & 5 & 3 \end{pmatrix}_{1\times 3} \qquad B = \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix}_{3\times 1}$$

$$A \cdot B = \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix} = 1 \cdot 7 + 5(-2) + 3(5)$$

$$= 7 - 10 + 15 = 12$$

$$A = \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{1\times 5} \end{array} \qquad B = \begin{array}{cc} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{5\times 1} & \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{array} \end{array}$$

$$A \cdot B = 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1$$

$$= 0 + 0 + 0 + 1 + 1 = 2$$

$$(a_i)_{1\times n} \cdot (b_j)_{n\times 1} = C_{1\times 1}$$

Requires n multiplications and n-1 additions. Since multiplications take more time, we simply count the number of multiplications to estimate the time complexity needed

Matrix multiplication

Suppose A is an mxp matrix and B a qxn matrix. The matrix multiplication AB is defined if p=q.

The result is a matrix of order mxn.

The (i,j)th entry of the result matrix is the dot product of row i of A and column j of B.

Let $A = (a_{ij})_{m \times \underline{p}}$ and $B = (b_{ij})_{\underline{p} \times n}$

If $C = A \cdot B$, Then $C = (c_{ij})_{m \times n}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}_{2\times 4} = \begin{pmatrix} a_{row1} \\ a_{row2} \end{pmatrix}_{2\times 4}$$

$$a_{row1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \\ b_{41} & b_{42} \end{pmatrix}_{4\times 2} = \begin{pmatrix} b_{col1} & b_{col2} \end{pmatrix}_{4\times 2} \quad b_{col1} = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{pmatrix} \quad b_{col2} = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{pmatrix}$$

Let $C = A \cdot B$, $\quad C = (c_{ij})_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$

$$C = A \cdot B = \begin{pmatrix} \text{Dot product of } a_{row1} \cdot b_{col1} & a_{row1} \cdot b_{col2} \\ a_{row2} \cdot b_{col1} & a_{row2} \cdot b_{col2} \end{pmatrix}$$

2/6/12

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}_{3\times 3} \qquad B = \begin{pmatrix} 2 & 4 \\ 1 & 1 \\ & 0 \end{pmatrix}_{3\times 2}$$

match ?

result size

AB defined ? Yes

Result $C = (c_{ij})_{3\times 2}$

$$\begin{array}{cc} & B \\ A & \boxed{\begin{matrix}2\\1\\3\end{matrix}}\ \boxed{\begin{matrix}4\\1\\0\end{matrix}} \\ \begin{matrix}\boxed{1\ \ 0\ \ 4}\\ \boxed{2\ \ 1\ \ 1}\\ \boxed{3\ \ 1\ \ 0}\end{matrix} & \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \end{array} = \begin{pmatrix} (1\ 0\ 4)\cdot\begin{pmatrix}2\\1\\3\end{pmatrix} & (1\ 0\ 4)\cdot\begin{pmatrix}4\\1\\0\end{pmatrix} \\ (2\ 1\ 1)\begin{pmatrix}2\\1\\3\end{pmatrix} & (2\ 1\ 1)\begin{pmatrix}4\\1\\0\end{pmatrix} \\ (3\ 1\ 1)\begin{pmatrix}2\\1\\3\end{pmatrix} & (3\ 1\ 1)\begin{pmatrix}4\\1\\0\end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 2 + 0\cdot 1 + 4\cdot 3 & 1\cdot 4 + 0\cdot 1 + 4\cdot 0 \\ 2\cdot 2 + 1\cdot 1 + 1\cdot 3 & 2\cdot 4 + 1\cdot 1 + 1\cdot 0 \\ 3\cdot 2 + 1\cdot 1 + 0\cdot 3 & 3\cdot 4 + 1\cdot 1 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \end{pmatrix}$$

In general,

$$A = (a_{ij})_{m\times p} \qquad B = (b_{ij})_{p\times n}, \quad C = A\cdot B = (c_{ij})_{m\times n}$$

Where $c_{ij} = (\text{row } i \text{ of } A)\cdot\begin{pmatrix}\text{col}\\ j.\\ \text{of}\\ B\end{pmatrix}$

$$c_{ij} = (a_{i1}\ a_{i2}\ \cdots\ a_{ip})\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}\cdot b_{pj}$$