Example:

$A = \begin{pmatrix} 1 & 5 & 4 & 7 \end{pmatrix}_{1\times 4}$ $B = \begin{pmatrix} 7 \\ -4 \\ 2 \\ 3 \end{pmatrix}_{4\times 1}$

A·B defined? yes — Scalar

B·A defined? yes → 4x4 matrix

$\underset{4\times 1}{B} \cdot \underset{1\times 4}{A}$

$$A \cdot B = \begin{pmatrix} 1 & 5 & 4 & 7 \end{pmatrix} \begin{pmatrix} 7 \\ -4 \\ 2 \\ 3 \end{pmatrix} = 1\cdot 7 + 5\cdot(-4) + 4\cdot 2 + 7\cdot 3 = 16$$

mults = 4

$B_{4\times 1} \cdot A_{1\times 4} = C_{4\times 4}$ # mults. $4\times 1\times 4 = 16$

$$\begin{array}{c|c} & \overbrace{[1]\ \ [5]\ \ [4]\ \ [7]}^{\text{Cols. of A}} \\ \hline \text{rows of B}\ \begin{array}{c}[7]\\ [-4] \\ [2] \\ [3]\end{array} & \begin{pmatrix} \underline{c_{11}} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \end{array} = \begin{pmatrix} 7\cdot 1 & 7\cdot 5 & 7\cdot 4 & 7\cdot 7 \\ -4\cdot 1 & -4\cdot 5 & -4\cdot 4 & -4\cdot 7 \\ 2\cdot 1 & 2\cdot 5 & 2\cdot 4 & 2\cdot 7 \\ 3\cdot 1 & 3\cdot 5 & 3\cdot 4 & 3\cdot 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 35 & 28 & 49 \\ -4 & -20 & -16 & -28 \\ 2 & 10 & 8 & 14 \\ 3 & 15 & 12 & 21 \end{pmatrix}_{4\times 4}$$

Result matrix size is determined by the number of rows of the first matrix and the number of columns of the second matrix.

# of multiplications = 16

$A = (a_{ij})_{m\times p}$ $B = (b_{ij})_{p\times n}$

$C = A\cdot B = (c_{ij})_{m\times n}$ # multiplications $= m\times p\times n$

Matrix multiplication algorithm

$$(a_{ij})_{m\times p} \qquad (b_{ij})_{p\times n}$$

$$A_{m\times p} \cdot B_{p\times n} = C_{m\times n}$$

for $i = 1$ to m {

  for $j = 1$ to n {

    $c_{i,j} = 0$

    for $k = 1$ to p {

      $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$

    }

  } // for j

} // for i

$m \times n \times p$ — # of k iterations; # of j iterations; # of i iterations

$c_{11} = 0 + a_{11} \cdot b_{11}$ $k=1$

$c_{11} = \;\; + a_{12} \cdot b_{21}$ $k=2$

$c_{11} = \;\; + a_{13} \cdot b_{31}$ $k=3$

$c_{11} = \;\; + a_{1p} \cdot b_{p1}$ $k=p$

$$\begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1p}} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} b_{11} & & \\ b_{21} & & \\ \vdots & & \\ \vdots & & \\ b_{p1} & & \end{pmatrix}$$

Comp. Complexity $= m \cdot n \cdot p$ multiplications

$m \cdot n \cdot (p-1)$ additions

$= O(mnp)$

If $m = p = n$, $O(n^3)$

Matrix multiplication chain

$$A_{m\times p} \cdot B_{p\times q} \cdot C_{q\times n} = D_{m\times n}$$

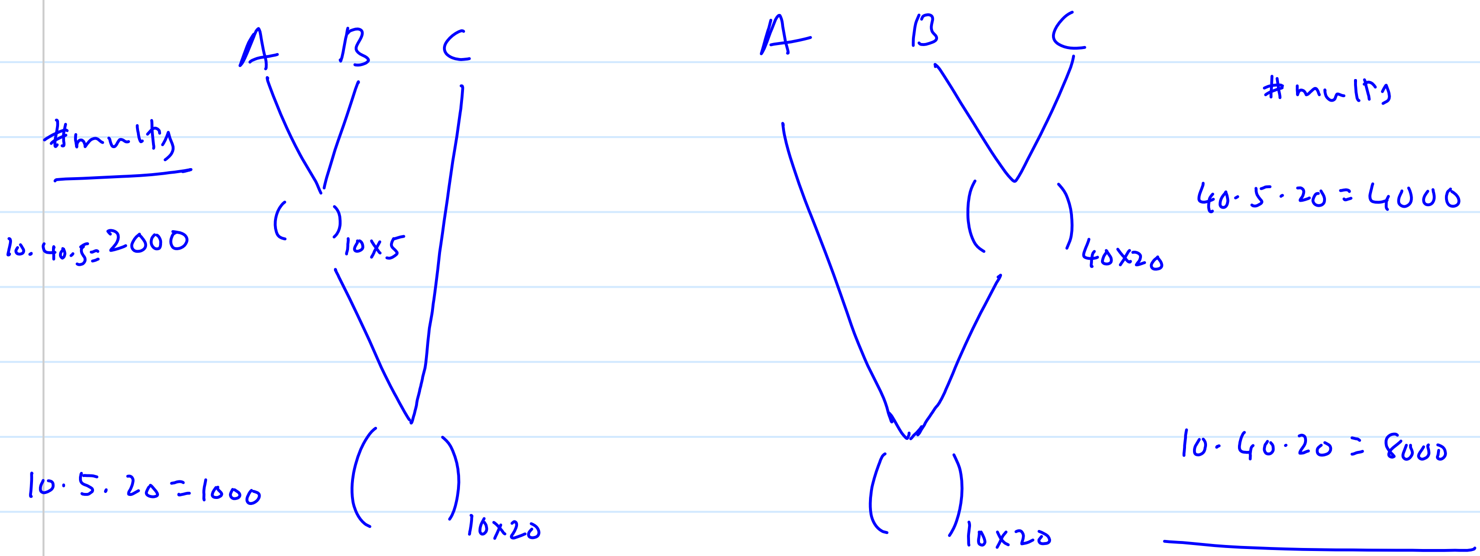
matrix mult. is associative

$$(A\cdot B)\cdot C = D \qquad\qquad A\cdot(B\cdot C) = D$$

Which is better in terms of computational time?

Calculate the complexity of each approach and pick the one with lesser complexity

$$A = (a_{ij})_{10\times 40} \qquad B = (b_{ij})_{40\times 5} \qquad C = (c_{ij})_{5\times 20}$$



3000 ⟵ Total # of multiplications ⟶ 12,000

∴ The order of multiplication can cut down computational time.

$$A_1 \cdot (A_2 \, (A_3 \cdot A_4)) \cdot (\cdot) \; A_n$$

Powers of a matrix

A is an $n \times n$ matrix.

Then $A^2 = \underbrace{A \cdot A}_{n \times n}$ is defined

$$A^3 = \underset{n\times n}{A} \cdot \underset{n\times n}{(A^2)} = (\quad)_{n \times n} = A \cdot A \cdot A$$

$$A^4 = A \cdot A^3 = A \cdot A \cdot A \cdot A$$

$\vdots$

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}$$

$$A^0 = I_n$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ & & \ddots & 0 \\ & 0 & & 1 \end{pmatrix}$$

Transpose of a matrix

Given a matrix A, its transpose it obtained by rewriting rows of A as columns. Transpose of A is denoted by A^t.

$$A = \begin{pmatrix} 1 & 0 & -4 & 5 \end{pmatrix} \quad A^t = \begin{pmatrix} 1 \\ 0 \\ -4 \\ 5 \end{pmatrix} \quad A \cdot A^t = 1^2 + 0^2 + (-4)^2 + 5^2 = 1 + 16 + 25 = 42$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2 \times 3} \quad A^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3 \times 2}$$

In general, $A = (a_{ij})_{m \times n}$, $A^t = (a_{ji})_{n \times m}$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\quad A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

For a square matrix, the main diagonal remains the same. The off-diagonal elements are moved to the other side of the diagonal.

If $A = A^t$, then A is a symmetric matrix.

If L is lower triangular matrix of size n×n, then L^t is an upper triangular matrix of size n×n and vice versa.

Determinant of a matrix

Deteminants are defined for square matrices.
The determinant of a square matrix of order n×n is a function that assigns a scalar value to each possible nxn matrix.

If $A = (a_{ij})_{n \times n}$, then |A| or det(A) denotes the determinant of A.

If |A| = 0, then A is a singular matrix.

<u>2x 2 matrix</u>

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

3x3 $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

<u>Sarrus's Rule</u>

$$|A| = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

$$- a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

Works only for 3x3 matrices.

Laplace Expansion

Each element of A, (a_{ij}), has a minor M_{ij} given by the determinant of the submatrix obtained by removing row i and column j of A.

$$A = (a_{ij})_{3\times3} \qquad M_{11} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

C_{ij}, Cofactor of $a_{ij} = (-1)^{i+j} M_{ij}$. The matrix of cofactors $C=(C_{ij})$ is the adjoint of A.

$$|A| = a_{11}\cdot C_{11} + a_{12}\cdot C_{12} + \cdots + a_{1n} C_{1n} \quad \text{Cofactors}$$

$n\times n$

$$= a_{11}(-1)^{1+1} M_{11} + a_{12}\cdot(-1)^{1+2}\cdot M_{12} + \cdots + a_{1n}(-1)^{1+n}\cdot M_{1n}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\cdot(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + a_{13}(-1)^{1+3} M_{13} + a_{14}(-1)^{1+4} M_{14}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14}$$

$$M_{11} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} & \cancel{a_{14}} \\ \cancel{a_{21}} & a_{22} & a_{23} & a_{24} \\ \cancel{a_{31}} & a_{32} & a_{33} & a_{34} \\ \cancel{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad M_{12} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} & \cancel{a_{14}} \\ a_{21} & \cancel{a_{22}} & a_{23} & a_{24} \\ a_{31} & \cancel{a_{32}} & a_{33} & a_{34} \\ a_{41} & \cancel{a_{42}} & a_{43} & a_{44} \end{vmatrix}$$

Examples

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad |A| = 1\cdot 4 - 2\cdot 3 = -2$$

$$B = \begin{pmatrix} 1 & 3 & \boxed{2} \\ 2 & 1 & 1 \\ 4 & 0 & 3 \end{pmatrix}, \quad |B| = ?$$

$$B = 1.\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= 1.(1\cdot 3 - 0\cdot 1) - 3(2\cdot 3 - 4\cdot 1) + 2(2\cdot 0 - 4\cdot 1)$$

$$= 3 - 6 - 8 = -11$$

det(A) = det(A^t)

Rank of a matrix

Let $A = (a_{ij})_{m \times n}$

Rank of A is the size of the largest square submatrix of A whose determinant is nonzero.

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Rank $(A) =$

For a diagonal matrix $D = \begin{pmatrix} d_{11} & 0 \cdots 0 \\ & d_{22} & 0 \\ 0 & & \ddots \\ & & & d_{nn} \end{pmatrix}$, $|D| = d_{11} \cdot d_{12} \cdots d_{nn}$

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$ Rank$(A) \neq 3$

a 2×2 Submatrix of A

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Another 2×2 submatrix of A

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$

Rank of A is 2. All you need to show is there is a 2×2 submatrix of A that is nonsingular -- determinant is nonzero.

Example: $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ Rank$(P) = 3$

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 0 \cdot 0 - 0 \cdot 0 + 1 \cdot (-1) = -1$$

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{pmatrix}_{3\times 4}$$

$\text{Rank}(A) \leq \min\{\#\text{rows}, \#\text{cols}\}$
$= \min\{3, 4\}$
$= 3$

non-singular $\quad \therefore \text{Rank}(A) = 3$

$$B = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 5 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{matrix} \end{pmatrix}_{5\times 6}$$

$\text{Rank}(B) = 5.$