2/8/12

Inverse of a matrix

Let A be a square matrix of size n.
If $|A| \neq 0$, then A is a non-singular matrix and there exists an $n \times n$ matrix, denoted A^{-1}, such that $A.A^{-1} = I_n$.

A^{-1} is unique.

For a 2x2 matrix, the inverse is calculated as follows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} \text{ exists if } |A| = ad - bc \neq 0.$$

Let us assume that $|A| \neq 0$. Then,

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Prove that $AA^{-1} = A^{-1}A = I_2$ and $(A^{-1})^{-1} = A$.

For larger square matrices, finding the inverse is significantly more complex.

Elementary row operations

1. Row switching $\quad$ Row $i \longleftrightarrow$ Row j

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xRightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

2. Row multiplication by a constant $\quad k \neq 0$

$k \cdot R_i \rightarrow R_i$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xRightarrow{2R_2 \rightarrow R_2} \begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix}$$

3. Replace a row by a sum of that row and a multiple of another row

$R_i + k \cdot R_j \rightarrow R_i \,, \quad k \neq 0$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xRightarrow{R_1 + R_2 \rightarrow R_1} \begin{pmatrix} 5 & 7 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Any nonsingular square matrix can be reduced to an identity matrix using elementary row operations.

Elementary column operations can be defined in a similar manner.

Gauss-Jordan elimination method to find the inverse of a matrix

$$A = (a_{ij})_{n \times n}$$

$$\underbrace{\begin{array}{cccc|cccc} & A & & & & I_n & & \\ a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & \cdots & 1 \end{array}}_{\text{augmented matrix}}$$

Use elementary row operations to reduce the left half of the augmented matrix to identity matrix.

The right half of the resulting augmented matrix is the inverse of the original matrix.

Example: $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Find A^{-1}. $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 2\cdot 4 - 1\cdot 3$

$= 5 \neq 0$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_1\cdot\frac{1}{2}\to R_1} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2-R_1\to R_2} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 1-1 & 4-\frac{3}{2} & 0-\frac{1}{2} & 1-0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{R_2\cdot\frac{2}{5}\to R_2} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2}\cdot\frac{2}{5} & -\frac{1}{2}\cdot\frac{2}{5} & 1\cdot\frac{2}{5} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right) \xrightarrow{R_1-\frac{3}{2}R_2\to R_1} \left(\begin{array}{cc|cc} 1-\frac{3}{2}\cdot 0 & \frac{3}{2}-1\cdot\frac{3}{2} & \frac{1}{2}-(-\frac{1}{5})\cdot\frac{3}{2} & 0-\frac{2}{5}\cdot\frac{3}{2} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2}+\frac{3}{10} & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right)$$

A^{-1}

Check:

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 2\cdot\frac{4}{5}+3\cdot(-\frac{1}{5}) & 2(-\frac{3}{5})+3\cdot\frac{2}{5} \\ 1\cdot\frac{4}{5}+4(-\frac{1}{5}) & 1(-\frac{3}{5})+4\cdot\frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{5}-\frac{3}{5} & -\frac{6}{5}+\frac{6}{5} \\ \frac{4}{5}-\frac{4}{5} & -\frac{3}{5}+\frac{8}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Gauss-Jordan Elimination Method

1. Augment the given square matrix $A_{n \times n}$ with I_n.

$$A \mid I$$

2. Replace Row1 by $\frac{1}{a_{11}} \cdot$ Row1.

3. Make all other entries in col 1 zeros by using the row op.

$$R_2 - a_{21} \cdot R_1 \longrightarrow R_2 \quad \text{for row2}$$
$$R_3 - a_{31} R_1 \longrightarrow R_3 \quad \text{for row3}$$
$$\vdots$$
$$R_n - a_{n1} R_1 \longrightarrow R_n \quad \text{for row } n$$

The augmented matrix is now of the form

$$\left[\begin{array}{ccccc|cccc} 1 & a'_{11} & a'_{12} & \cdots & a'_{1n} & \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & -\frac{a_{21}}{a_{11}} & \cdots & & \\ \vdots & & & & & \vdots & & & \\ 0 & a'_{n2} & a'_{n3} & \cdots & a'_{nn} & -\frac{a_{n1}}{a_{11}} & \cdots & & \end{array}\right]$$

4. Repeat 2 & 3 for other rows & columns.

If the diagonal element of the current row, the element (i,i) position in iteration i, is 0, switch that row with another row that has nonzero value in that column position. For invertible matrices, this is always feasible.

This switching of rows to ensure the diagonal element is nonzero is called **pivoting**.

Solution to system of Linear equations

$$x_1 + 2x_2 + x_3 = 4$$

$$3x_1 - 4x_2 + 2x_3 = 2$$

$$5x_1 + 3x_2 + 5x_3 = -1$$

$$\underbrace{\begin{pmatrix} \boxed{1 \quad 2 \quad 1} \\ 3 & -4 & 2 \\ 5 & 3 & 5 \end{pmatrix}}_{A_{3\times3}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x_{3\times1}} = \underbrace{\begin{pmatrix} \boxed{4} \\ 2 \\ -1 \end{pmatrix}}_{b_{3\times1}} \qquad 1\cdot x_1 + 2\cdot x_2 + 1\cdot x_3 = 4$$

$$Ax = b$$

If A is non-singular, $|A| \neq 0$, then A^{-1} exists.

$$\Rightarrow \quad A^{-1}Ax = A^{-1}b$$

$$I \cdot x = A^{-1}b \quad \Rightarrow \quad x = A^{-1}b.$$

Note: If the equations are independent, that is, none of the equations can be obtained by a linear combination of the other two equations, then the corresponding matrix is nonsingular.

Gauss-Jordan Elimination to Solve System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{b}$$

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad \text{or} \qquad x = A^{-1}b$$

Augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \xrightarrow[\text{row ops}]{\text{use elementary}} \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1' \\ 0 & 1 & 0 \cdots & 0 & b_2' \\ & & \ddots & 0 & \vdots \\ 0 & & & 1 & b_n' \end{array}\right)$$

$$\underbrace{\begin{matrix} b_1' \\ b_2' \\ \vdots \\ b_n' \end{matrix}}_{A^{-1}b}$$

Example:

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \\ 5 & 3 & 5 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}}_{b}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & -4 & 2 & 2 \\ 5 & 3 & 5 & -1 \end{array}\right) \xrightarrow{R_2 - 3R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3-3\cdot 1 & -4-3\cdot 2 & 2-3\cdot 1 & 2-3\cdot 4 \\ 5 & 3 & 5 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -10 & -1 & -10 \\ 5 & 3 & 5 & -1 \end{array}\right) \xrightarrow{R_3 - 5R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -10 & -1 & -10 \\ 5-5\cdot 1 & 3-5\cdot 2 & 5-5\cdot 1 & -1-5\cdot 4 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -10 & -1 & -10 \\ 0 & -7 & 0 & -21 \end{array}\right) \xrightarrow{\frac{-R_2}{10} \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & -7 & 0 & -21 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & -7 & 0 & 21 \end{array}\right) \xrightarrow[R_3 + 7R_2 \to R_3]{R_1 - 2R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 2-2 & 1-\frac{2}{10} & 4-2 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & -7+7 & 0+\frac{7}{10} & -21+7 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{8}{10} & 2 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & 0 & \frac{7}{10} & -14 \end{array}\right) \xrightarrow{\frac{10}{7}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{8}{10} & 2 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & 0 & 1 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{8}{10} & 2 \\ 0 & 1 & \frac{1}{10} & 1 \\ 0 & 0 & 1 & -20 \end{array}\right) \xrightarrow[R_2 - \frac{1}{10}R_3 \to R_2]{R_1 - \frac{8}{10}R_3 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2-(-20)\cdot\frac{8}{10} \\ 0 & 1 & 0 & 1-(-20)\frac{1}{10} \\ 0 & 0 & 1 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -20 \end{array}\right)$$

$\therefore$ $x_1 = 18$, $x_2 = 3$, $x_3 = -20$

will be the solution.

check:

$$\underset{A}{\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \\ 5 & 3 & 5 \end{pmatrix}} \underset{x}{\begin{pmatrix} 18 \\ 3 \\ -20 \end{pmatrix}} = \begin{pmatrix} 1\cdot18 + 2\cdot3 - 20\cdot1 \\ 3\cdot18 - 4\cdot3 + 2\cdot(-20) \\ 5\cdot18 + 3\cdot3 - 5\cdot20 \end{pmatrix} = \underset{b}{\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}}$$

Problem 5, Section 2.6

$$\overset{B}{\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}} \cdot A = \overset{C}{\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}} \qquad \text{Solve for } A.$$

$$\underbrace{B^{-1} \cdot B}A = B^{-1} C \qquad |B| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 2 \cdot 4 - 1 \cdot 3$$

$$\underbrace{I_2 \cdot A} = B^{-1} C \qquad = 5$$

$$A = B^{-1} C \qquad \Rightarrow B^{-1} \text{ exists.}$$

$$\text{Adj of } B = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{|B|} \cdot (\text{Adj } B)^t = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{-3}{5} \\ \frac{-1}{5} & 2/5 \end{pmatrix}$$

$$A = B^{-1} C = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} [4 \ \ -3] \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} & [4 \ \ -3] \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ [-1 \ \ 2] \begin{bmatrix} 3 \\ 1 \end{bmatrix} & [-1 \ \ 2] \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 9 & -6 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} & \frac{-6}{5} \\ \frac{-1}{5} & \frac{4}{5} \end{pmatrix}$$

14.

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ & \ddots & \ddots & 0 \\ \mathbf{0} & & & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & & & \mathbf{0} \\ & b_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & & & \mathbf{0} \\ & a_{22}b_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & a_{nn}b_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{11}b_{13} \\ 0 & a_{22}b_{22} & 0 \\ a_{33}b_{31} & 0 & 0 \end{pmatrix}$$

15.

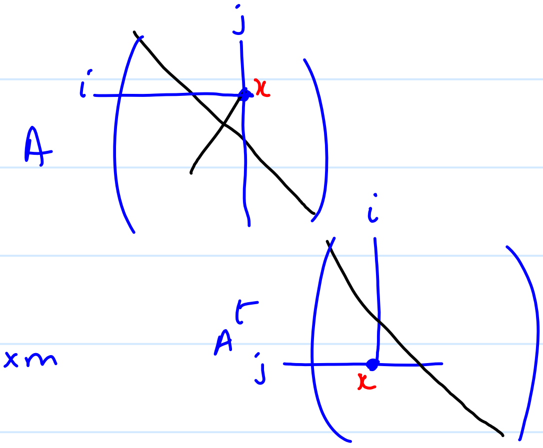
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} \boxed{1 \quad 1} \\ \boxed{0 \quad 1} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 1\cdot 0 & 1\cdot 1 + 1\cdot 1 \\ 0\cdot 1 + 1\cdot 0 & 0\cdot 1 + 1\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A \cdot A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad n \geq 0$$

$$A^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



16. Show that $(A^t)^t = A$.

$$A = (a_{ij})_{m \times n} \qquad A^t = (a_{ji})_{n \times m}$$

$$(A^t)^t = (a_{ij})_{m \times n}$$

Show that $(AB)^{-1} = B^{-1}A^{-1}$, where A and B are nonsingular n×n matrices.

$$(AB)\cdot(\underline{B^{-1}A^{-1}}) = A(BB^{-1})A^{-1} = (A\cdot I_n)A^{-1} = AA^{-1} = I_n.$$

$\therefore$ AB is inverse of $B^{-1}A^{-1}$.

$\therefore (AB)^{-1} = B^{-1}A^{-1}$.

In general, $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}$,

where $A_1, \ldots, A_k$ are invertible nxn matrices.

$\therefore (ABC)^{-1} = C^{-1}B^{-1}A^{-1}, \quad (ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}, \ldots$

21. Show that $(A^n)^{-1} = (A^{-1})^n$, where A is a nonsingular nxn matrix.

$$\begin{aligned}(A^n)^{-1} = (A\,A^{n-1})^{-1} &= (A^{n-1})^{-1}\cdot A^{-1} \\ &= (A\,A^{n-2})^{-1}\,A^{-1} \\ &= (A^{n-2})^{-1}A^{-1}A^{-1} \\ &\;\vdots \\ &= \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n\text{ times}} = (A^{-1})^n\end{aligned}$$

22. Show that $(AA^t)^t = AA^t$, where A is an n×n matrix. [We will use the result from Problem 17b, $(AB)^t = B^t A^t$]

$$(A\underbrace{A^t}_{B})^t = (AB)^t = B^t A^t = (A^t)^t A^t = AA^t$$