12a. Show that (A + B)C = AC + BC, where A, B and C are matrices and the sum A+B and products AC and BC are defined.

LHS: Let $(A+B)\cdot C = \underline{D}$ $\qquad d_{ij} = (\text{Row } i \text{ of } A+B)\cdot\begin{pmatrix}\text{col.}\\ j\\ \text{of}\\ C\end{pmatrix}$

$A = (a_{ij})_{m\times p} \qquad B = (b_{ij})_{m\times p}, \qquad C = (c_{ij})_{p\times n}$

$$\text{Row } i \text{ of } A+B = (\text{row } i \text{ of } A) + (\text{row } i \text{ of } B)$$

$$= (a_{i1}\ a_{i2}\ \cdots\ a_{ip}) + (b_{i1}\ b_{i2}\ \cdots\ b_{ip})$$

$$= (a_{i1}+b_{i1}\quad a_{i2}+b_{i2}\quad \cdots \quad a_{ip}+b_{ip})$$

$$d_{ij} = (a_{i1}+b_{i1}\quad a_{i2}+b_{i2}\quad \cdots\quad a_{ip}+b_{ip})\begin{pmatrix}c_{1j}\\ c_{2j}\\ \vdots\\ c_{pj}\end{pmatrix}$$

$$= (a_{i1}+b_{i1})\cdot c_{1j} + (a_{i2}+b_{i2})\cdot c_{2j} + \cdots + (a_{ip}+b_{ip})\cdot c_{pj}$$

$$= a_{i1}\cdot c_{1j} + b_{i1}\cdot c_{1j} + a_{i2}\cdot c_{2j} + b_{i2}\cdot c_{2j} + \cdots + a_{ip}c_{pj} + b_{ip}c_{pj}$$

$$= a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{ip}c_{pj}$$
$$+ b_{i1}c_{1j} + b_{i2}c_{2j} + \cdots + b_{ip}c_{pj}$$

$$= (\text{row } i \text{ of } A)\cdot\begin{pmatrix}\text{col } j\\ \text{of}\\ C\end{pmatrix} + (\text{row } i \text{ of } B)\cdot\begin{pmatrix}\text{col } j\\ \text{of } C\end{pmatrix}$$

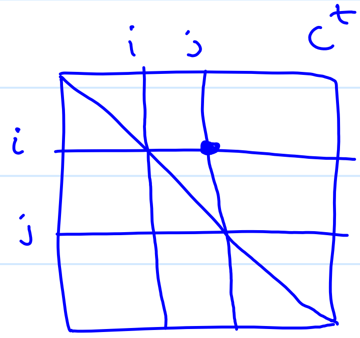
Each dij can be rewriten as the sum of the dot produts of row i of A with column j of C and row i of B with column j of C.

$\therefore \quad D = AC + BC$

17. Show that (a) $(A+B)^t = A^t+B^t$ and (b) $(AB)^t = B^tA^t$, where A and B are n×n matrices.

Let $AB = C$ $\quad A = (a_{ij})_{n\times n}$ $\quad B = (b_{ij})_{n\times n}$ $\quad C = (c_{ij})_{n\times n}$

$(AB)^t = C^t$



What is the (i,j)th element of C^t

$= (j,i)$th element of C

$= (j\text{th row of } A) \cdot \begin{pmatrix} i\text{th} \\ \text{col} \\ \text{of } B \end{pmatrix}$

$$= (a_{j1}\ a_{j2} \cdots a_{jn}) \begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix} = a_{j1} \cdot b_{1i} + \cdots + a_{jn} b_{nj}$$

$$= \sum_{k=1}^{n} a_{jk} \cdot b_{ki}$$

Let $B^t \cdot A^t = D = (d_{ij})_{n\times n}$

$d_{ij} = [i\text{th row of } B^t] \cdot \begin{bmatrix} j\text{th col. of} \\ A^t \end{bmatrix}$

$= [b_{1i} \cdots b_{ni}] \begin{bmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{bmatrix}$

$= b_{1i} a_{j1} + \cdots + b_{ni} a_{jn}$

$= a_{j1} b_{1i} + \cdots + a_{jn} b_{ni}$

$= (i,j)$th element of C^t

$\therefore (AB)^t = B^t A^t$.

Properties of inverses

$AA^{-1} = A^{-1}A = I_n$, $A = (\ )_{n\times n}$ $|A| \neq 0$

$(A^{-1})^{-1} = A$

$(k \cdot A)^{-1} = \frac{1}{k} \cdot A^{-1}$, $k \neq 0$ scalar

$(A^t)^{-1} = (A^{-1})^t$

$(AB)^{-1} = B^{-1}A^{-1}$. A, B are nonsingular $n\times n$

Properties of Determinants

$$|A| = |A^t|$$

EROs (elementary row operations). Let $A = (a_{ij})_{n\times n}$

Let $A \xrightarrow{ERO} B$.

(i) $A \xrightarrow{R_i \leftrightarrow R_j} B$

$|A| = -|B|$

(ii) $A \xrightarrow{kR_i \to R_i} B$, $k \neq 0$

$|B| = k \cdot |A|$

(iii) $A \xrightarrow{kR_i + R_j \to R_j} B$

$|A| = |B|$

$$|I_n| = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & O & \ddots & 1 \end{vmatrix} = 1$$

If A is a diagonal, UT, or LT $n\times n$ matrix, then

$$|A| = a_{11} a_{22} \cdots a_{nn}.$$

$$|A+B| = |A| + |B|$$

$|AB| = |A| \cdot |B|$, A & B are square matrices $n\times n$

$$|kA| = k^n |A|$$

$|A^{-1}| = \dfrac{1}{|A|}$ $\qquad |A|\,|A^{-1}| = 1$

Eigenvalues and Eigenvectors

Consider an equation of the form **Ax** = λ**x**, where **A**=$(a_{i,j})_{n\times n}$ a matrix of knowns, **x**=$(x_i)_{n\times 1}$ a vector of unkowns, and λ is an unkown scalar.

if $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \lambda x = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$

If the equation is satisfied for x other than the null vector, then each such x is an eigenvector,

also called characteristic vector of A, and λ is an eigenvalue.

$Ax = \lambda x$

$Ax - \lambda x = 0 \leftarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Note: $Ix = x$

$Ax - \lambda \cdot I \cdot x = 0$

$(A - \lambda I) \cdot x = 0$

$x = 0$ will satisfy it

For nonnull solutions of x, find λ s.t. $\underbrace{|A - \lambda I|}_{\text{Characteristic equation of } A} = 0$

Let $A = (a_{ij})_{3\times 3}$. $\quad A - \lambda I = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix}$$

Solve $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$ for λ and use $Ax = \lambda x$ to find x vectors.

Example Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Solve

$$A - \lambda I = \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 3\cdot 1 = 0 \quad \Rightarrow \quad 8 - 6\lambda + \lambda^2 - 3 = 0$$

$$\Rightarrow \boxed{\lambda^2 - 6\lambda + 5 = 0} \leftarrow \text{ch. equation of } A.$$

$$\Rightarrow (\lambda - 5)(\lambda - 1) = 0$$

$\Rightarrow \lambda = 1$ or 5 are the solutions.

$\Rightarrow$ 1 and 5 are the eigenvalues of A.

$\lambda = 1$ $\qquad A x = x$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{——— (1)}$$

$$4x_1 + x_2 = x_1 \Rightarrow 3x_1 + x_2 = 0 \Rightarrow x_2 = -3x_1.$$

$$3x_1 + 2x_2 = x_2 \Rightarrow 3x_1 + x_2 = 0$$

$\therefore$ any vector of the form $\begin{pmatrix} k \\ -3k \end{pmatrix}$, k is a scalar, satisfies (1). For $\lambda = 1$ all vectors of the form $\begin{pmatrix} k \\ -3k \end{pmatrix}$ are the corresponding eigenvectors.

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 17 \\ -51 \end{pmatrix}, \begin{pmatrix} 1/3 \\ -1 \end{pmatrix}, \ldots$$

$\lambda = 5$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$$

$4x_1 + x_2 = 5x_1 \Rightarrow \quad x_2 = x_1$

$3x_1 + 2x_2 = 5x_2 \Rightarrow \quad 2x_2 = 2x_1 \quad$ or $\quad x_2 = x_1$.

Vectors of the form $\begin{pmatrix} k \\ k \end{pmatrix}$ are eigenvectors of A corresponding to eigenvalue $\lambda = 5$.

Trace of a matrix tr(A) $\quad tr(A) = a_{11} + a_{22} + \cdots + a_{nn} \quad A = (a_{ij})_{n \times n}$

Sum of all eigenvalues of A = $tr(A)$.

$A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, $\quad tr(A) = 4 + 2 = 6$

eigenvalues $\lambda = 1$ or 5. $\quad$ Sum $1 + 5 = 6$

Product of eigenvalues

product of eigenvalues $= |A|$

$1 \cdot 5 = 5$

$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - 3 \cdot 1 = 5$

Extra: not covered in class

9. Show that A + (B+C) = (A+B) + C, where A, B and C are matrices of order m×n.

$$A = (a_{ij})_{m\times n} \qquad B = (b_{ij})_{m\times n} \qquad C = (c_{ij})_{m\times n}$$

LHS :

$$B+C = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} + \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \\ c_{m1} & \cdots & c_{mn} \end{pmatrix} = \begin{pmatrix} b_{11}+c_{11} & \cdots & b_{1n}+c_{1n} \\ \vdots & & \\ b_{m1}+c_{m1} & \cdots & b_{mn}+c_{mn} \end{pmatrix}$$

$$B+C = (b_{ij})_{m\times n} + (c_{ij})_{m\times n} = (b_{ij}+c_{ij})_{m\times n}$$

$$A+(B+C) = (a_{ij})_{m\times n} + (b_{ij}+c_{ij})_{m\times n} = (a_{ij}+(b_{ij}+c_{ij}))_{m\times n}$$

$$= (a_{ij}+b_{ij}+c_{ij})_{m\times n}$$

$$= ((a_{ij}+b_{ij})+c_{ij})_{m\times n}$$

$$= (a_{ij}+b_{ij})_{m\times n} + (c_{ij})_{m\times n} = (A+B)+C \leftarrow RHS$$

11. If AB and BA are defined, what can you say regarding the sizes of A and B?

$$A = (\quad)_{m\times p} \qquad B = (\quad)_{q\times n}$$

If AB is defined $p=q$ If BA is defined $m=n$

$$\therefore A = (\quad)_{m\times p}, \quad B = (\quad)_{p\times m}$$