Combinatorics

2/22/12

Note Title

Basic Counting Principles [KR, Section 6.1]

Product Rule

If a task can be subdivided (or broken) into a sequence of two subtasks such that the first subtask can be done in one of n_1 ways and the second subtask can be done in one of n_2 ways after the first task is done, then the total number of ways to do the original task is $n_1 * n_2$.

Example 0: What is the number of ways to have a lunch if a lunch consists of a sandwich and a drink? There are five distict types of sandwiches and three distinct types of drinks.

$$\overbrace{\text{Lunch}}^{\text{Task}} = \overbrace{\text{1 Sandwich}}^{\text{Subtask 1}} \ \& \ \overbrace{\text{1 drink}}^{\text{Subtask 2}}$$

$= 5 * 3 = 15$ ways.

Example 2: Chairs are labeled with a letter followed by a number in the range 1 to 100. How many Chairs can be labeled distinctly? Or equivalently, how many ways are there to label a chair?

Task: Label Chair

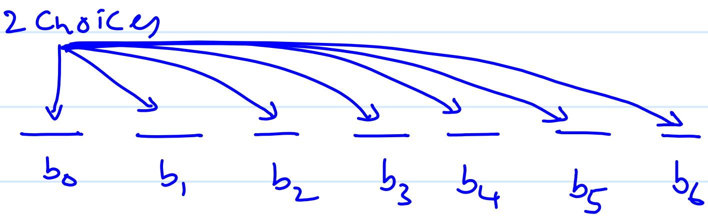
Choose a letter — Choose a number in [1, 100]

Letter	Number
26 choices	100 choices

26 * 100 = 2600 ways

$\therefore$ up to 2600 chairs can be labeled distinctly.

Example 4: How many distinct 7-bit strings are possible?



7 Subtasks.

Subtask #0: # b_0 choices $= 2$

Subtask #1 : # b_1 choices $= 2$

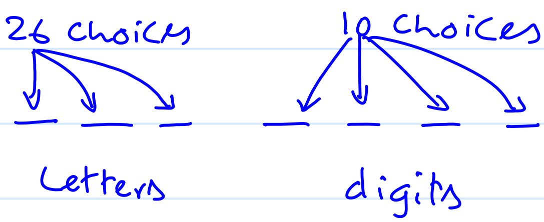
⋮

Subtask #6 : # b_6 choices $= 2$

∴ Total # of ways to form a 7-bit string

$$= \underbrace{2 \times 2 \times \cdots \times 2}_{(7 \text{ times})} = 2^7 = 128$$

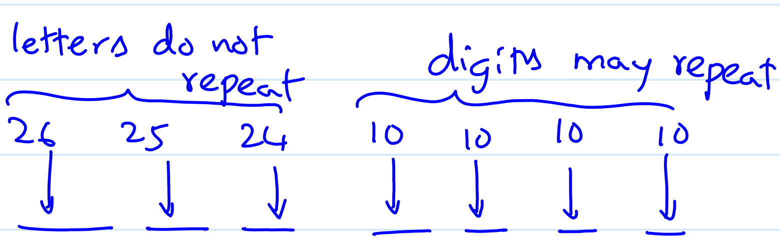
Example 5: License plates have 3 letters followed by 4 digits. What is the total number of distinct license plates?



Distinct license plates

$$= 26^3 \times 10^4$$

Example 5': Suppose the letters may not be repeated in the above problem. Recalculate the number of distinct license plates possible.



$$26 \times 25 \times 24 \times 10^4$$

Example 9: Calculate the value of k after executing the following pseudo codes.

```
k =0;
for i = 1 to 10 {
        for j = 1 to 20 {
                k++;
        }
}
```

k = 10*20

```
k=0;
for i_1 = 1 to n_1 {
   for i_2 = 1 to n_2 {
      ...
         for i_p = 1 to n_p {
              k++;
         }
      ...
   }
}
```

$k = n_1 * n_2 * \ldots * n_p$

```
k =0;
for i = 1 to 10 {
        for j = i to 20 {    // j varies from i to 20
                k++;
        }
}
```

(In the second listing, the **i** in "for j = **i** to 20" is printed in bold.)

Value of i	1	2	3	. . .	10
# j-loop iterations	20	19	18	. . .	11
range for j	1-20	2-20	3-20		10-20

$\therefore$ Total # iterations $= 20 + 19 + 18 + \cdots + 11$

$$= \frac{20+11}{2} * 10 = 31 * 5 = 155$$

Sum Rule

If a task can be done in one of n_1 ways or in one of n_2 ways that are different from the first set of n_1 ways, then the task can be done in $n_1 + n_2$ ways.

Example: JPL Cafeteria has 3 lunch packages
UC Cafeteria has 10 lunch choices.

choices to have a lunch from JPL or UC is $3+10=13$

Example 14: Calculate the value of k after executing the following pseudo code.

```
k=0;
for i1 = 1 to n1 {
        k++;
}
for i2 = 1 to n2 {
        k++;
}
...
for ip = 1 to np {
        k++;
}
```

$k = n_1 + n_2 + \ldots + n_p$

Example 0': What is the number of ways to have a lunch if a lunch consists of a sandwich and a drink? There are five distict types of sandwiches and three distinct types of drinks.

This problem can also be solved by the Sum Rule. In fact, the Product Rule can be considered as a special case shortcut for the Sum Rule.

Suppose the three choices for a drink are coffee, soda and juice.

Then a lunch consists of

coffee + a sandwich 5
+
or soda + a sandwich 5
+
or juice + a sandwich 5

15

Example 16': The password for a computer account can be 6, 7 or 8 characters in length; the characters can be letters or digits. What is the number of ways a password can be formed?

password

6 chars   7 chars   8 chars

passwords P_6 P_7 P_8

By Sum rule: Total # of passwords $= P_6 + P_7 + P_8$

26 letters & 10 digits = 36 choices

$\underline{P_6} = 36^6$

36 choices

_ _ _ _ _ _

$P_7 = 36^7 \quad P_8 = 36^8$

$\therefore$ Total number of distinct passwords $= P_6 + P_7 + P_8$

$= 36^6 + 36^7 + 36^8$

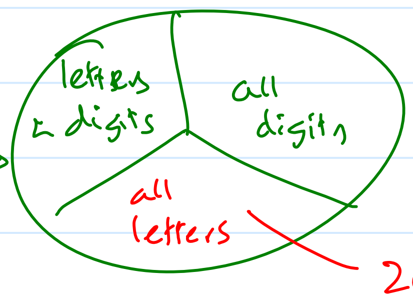
Example 16: If a password must have at least one digit to be valid, what is the number of valid passwords in the above problem?

6-char passwords $= 36^6$

includes passwords with letters only

invalid

36^6



26^6

VP_6 = valid 6-character passwords

6 char pws with at least one digit $= 36^6 - 26^6$

$\therefore$ Total # of words $= \underbrace{(36^6 - 26^6)}_{VP_6} + \underbrace{(36^7 - 26^7)}_{VP_7} + \underbrace{(36^8 - 26^8)}_{VP_8}$

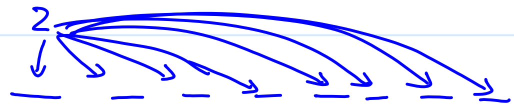
Principle of Inclusion-Exclusion (PIE)
(A more general version of the Sum rule. Also called the subtraction rule.)

If a task can be done in one of n_1 ways or in one of n_2 ways, but some n_3 ways of the set of n_1 ways are in common with the set of n_2 ways, then the total number of ways to do the task is $n_1 + n_2 - n_3$.

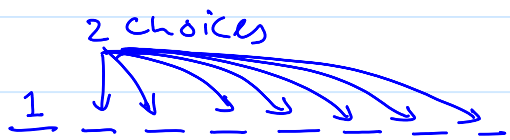
$(n_3 \le n_1, \ n_3 \le n_2)$

Example 18
How many bit strings of length eight start with an 1 or end with 00?

Total # of 8-bit strings $= 2^8$



Strings that start with 1



such strings $= \underbrace{2 \times 2 \times \cdots \times 2}_{7 \text{ times}} = 2^7$ n_1

Strings that end with 00

such strings $= \underbrace{2 \times 2 \times \cdots \times 2}_{6 \text{ times}} = 2^6$ n_2

Strings that start with 1 and end with 00 are common to both groups

$1 \ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5} \ 0 \ 0$

How many such strings? 2^5 n_3

The # of strings that start with '1' or end with '00'

$$= n_1 + n_2 - n_3 = 2^7 + 2^6 - 2^5.$$

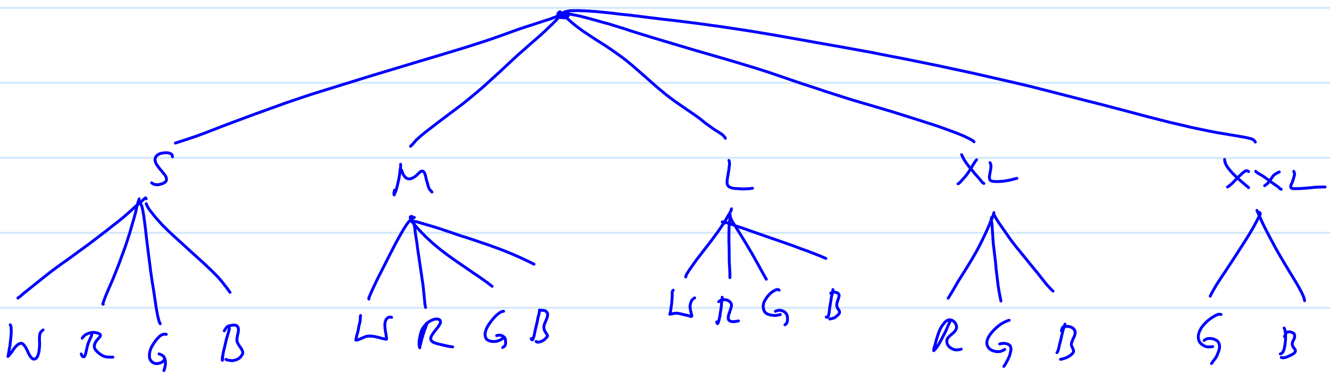
Example 23': Calculate the number of different t-shirts possible if the t-shirts are available in five sizes -- S, M, L, XL and XXL -- and each size is available in four colors -- white, red, green and black.

Simple application of product rule:
Number of different t-shirts = 5 * 4 = 20

Example 23: In the above problem, suppose XL comes in only red, green and black and XXL comes in only green and black. Recalculate the number of different t-shirts possible.

$n_S, n_M, n_L, n_{XL}, n_{XXL}$ – # different t-shirts in size S, M, L, XL, XXL.

$$\# \text{ T-shirts} = n_S + n_M + n_L + n_{XL} + n_{XXL}$$
$$= 4 + 4 + 4 + 3 + 2$$
$$= 17$$



Division Rule

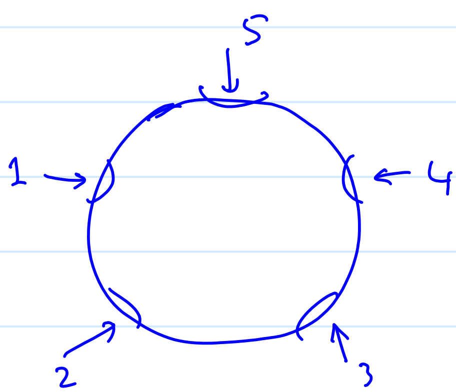
If n ways are feasible to do a task, but each of the n ways is the same as any of d ways based on a counting criterion. then there are only n/d ways to the task.

A rule to ignore "unimportant" differences when counting things.

Example 0'': There are 15 different sandwich and drink combinations for a lunch. If the choice of a drink does not matter, and there are three drink choices, what is the number of ways to pick a lunch?

Answer: By the division rule, 15/3 = 5. Why?

The original count considered different drink choices as distinct lunch combinations. Since there are three drink choices, for each sandwich choice, there must have been three combinations that had the sandwich and differed only in the drink choices. So the original count is higher by a factor of 3. Dividing the original count by 3 gives the numer of lunch combinations that differ in the type of sandwich chosen.



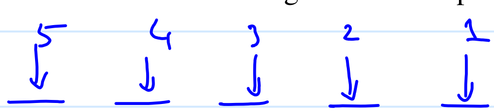
5 people to be seated around a round table in

$$5 \times 4 \times 3 \times 2 \times 1 = 5! \text{ ways}$$

However, if the table with people seated around it can be rotated so that a selected person is always in the north-most position, then there are only

$$\frac{5!}{5} = 4! \text{ ways}$$

Seating 5 people in a line. If the left-to-right order is important, then there are 5! ways form a line.



$$= 5 \times 4 \times 3 \times 2 \times 1.$$

However, if only the placements that result in distinct adjacencies of people are considered, then

$$P_1 \; P_2 \; P_3 \; P_4 \; P_5 = P_5 \; P_4 \; P_3 \; P_2 \; P_1$$

For each placement of people in a line in the original count, there is an equivalent right-to-left placement that has the same adjacencies among people. So the new count is

$$\frac{5!}{2}$$