Note Title

- Flock of 20 pigeons
- 19 pigeonholes to house them

$\Rightarrow$ At least one of the pigeonholes has at least two pigeons.

Theorem 1. If k is a positive integer and k+1or more objects are to be placed in k boxes, then at least one box contains at least two objects.

Proof: We prove it by contradiction.

That means, we assume that there are k+1 objects placed in k boxes, but none of the boxes have two or more objects.

$\Rightarrow$ Each box contains 0 or 1 objects.

$\therefore$ # objects in the k boxes $\leq \underbrace{1+1+\cdots+1}_{k \text{ times}} = k$

This contradicts the fact that k+1 objects are placed in these boxes.

$\therefore$ The assumption is incorrect.

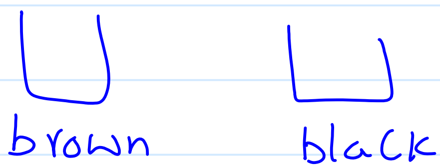
$\therefore$ The theorem statement is correct.

Example 3
The possible scores in a test are 0, 1, ..., 100. What is the minimum number of students that must take the test to guarantee that at least two students have the same score?

101 $\overbrace{\text{pigeonholes}}^{\text{test score possibilities}}$ $\rightarrow$ 102 $\overbrace{\text{pigeons}}^{\text{students}}$

Problem 3a, Section 6.2
A drawer has a dozen brown socks and a dozen black socks, randomly placed. If you draw socks at random w/o looking at the drawer, how many socks do you need draw to have a pair of socks of the same color?

2 "pigeonholes"



$2+1 = 3$ socks must be drawn

Problem 3b. How many socks must be taken out to ensure at least two white socks?

Worst case scenario: take out all brown socks first before taking out any white sock.
Taking out 14 ensures that there are at least two white socks.

Problem 5
In any group of five integers (not necessarily consecutive), at least two of them have the same remainder when divided by 4.

possible distinct remainders = 0, 1, 2, or 3
\# remainders = 4 $\leftarrow$ # pigeonholes.

a_1, a_2, a_3, a_4, a_5

$\left\lceil \frac{5}{4} \right\rceil = 2$ #s have the same remainder.

Generalized Pigeonhole Principle

If N, $N \geq 0$, objects are placed in k, $k \geq 1$, boxes, then at least one of the boxes has at least $\lceil N/k \rceil$ objects.

Proof: By contradiction. Assume that N objects are placed in k boxes, but that each and every box has fewer than $\lceil N/k \rceil$ objects.

$\therefore$ By assumption, each box has at most $\lceil N/k \rceil$-1 objects.

Total objects in k boxes $\leq k\,(\lceil N/k \rceil - 1)$.

Since $\lceil N/k \rceil < N/k + 1$ for all positive integer values of N and k,

$$k\,(\lceil N/k \rceil - 1) < k\,(N/k + 1 - 1) = k\,(N/k) = N$$

So, Total objects in k boxes $\leq k\,(\lceil N/k \rceil - 1) < N$

This is a contradiction since we started with the fact that N objects are placed in k boxes.

Hence the theorem.

Note: $\left\lceil \frac{N}{k} \right\rceil < \frac{N}{k} + 1$

for $N \geq 0$ and $k \geq 1$.

Try: $N = 15$, $k = 3$

$\left\lceil \frac{N}{k} \right\rceil = 5 < \frac{N}{k} + 1 = 6$

$N = 16$: $\left\lceil \frac{N}{k} \right\rceil = 6 < \frac{N}{k} + 1 = 6.33$

Example 5. In any group of 100 people, at least $\lceil 100/12 \rceil = 9$ are born in the same month.

Example 6'

What is the minimum number of students that must take the test to ensure that at least three students receive the same grade? A grade can be A, B, C, D or F.

The max number of students that take the test and not have three or more students with the same grade can be calcuclated as

# of grade levels * (the number of required -1)

Adding just one more student ensures that there are at least six students receiving the same grade.

$\boxed{2}$ A $\boxed{2}$ B $\boxed{2}$ C $\boxed{2}$ D $\boxed{2}$ F

$(3-1) * 5 + 1 = 11$

verify: $\left\lceil \frac{11}{5} \right\rceil = \lceil 2.2 \rceil = 3$

$\left\lceil \frac{N}{k} \right\rceil = 3 \Rightarrow \left\lceil \frac{N}{5} \right\rceil = 3$

$N = 10$ $\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{10}{5} \right\rceil = 2$

$N = 11$ $\left\lceil \frac{11}{5} \right\rceil = \lceil 2.2 \rceil = 3$

Problem 13a

If five integers are selected from the first eight positive integers, then there must be a pair of integers with their sum equal to 9.

1 8

2 7

3 6

4 5

$\boxed{8}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$

19. Discrete math class - 25 students

- freshman
- sophomore
- junior

(a) Show that there are either at least 9 freshmen, 9 sophomores, or 9 juniors.

By pigeonhole principle, at least $\left\lceil \frac{25}{3} \right\rceil = \lceil 8.3 \rceil = 9$ in at least one of the class levels.

(b) Show that there are either at least three freshman, at least 19 sophomores, or at least five juniors in the class.

Assume that the statement is false.

$\therefore$ There are at most 2 freshman, at most 18 sophomores, and at most 4 juniors.

$\therefore$ The total # of students by this assumption is at most $2 + 18 + 4 = 24$

a contradiction since the class has 25 students.

$\therefore$ The original statement must be true.

37, Section 6.2
Consider a network of six computers. Each computer is directly connected to zero or more of other computers. Show that there are at least two computers that have the same number of direct connections.

possible # of connections are 0, 1, 2, 3, 4, 5.

However, it is not feasible to one compute with 0 conn.

and another with the max. possible 5 connections.

conn. to the computers is 0, 1, 2, 3, or 4

or

1, 2, 3, 4, or 5

5 possibilities
6 comp.

$\lceil \frac{6}{5} \rceil = 2$ comp. with the same # of conn.

44. There are 51 houses on a street. Address range is 1000 to 1099.
Show that at least two houses have addresses that are consecutive integers.

Group an even addr followed by the next odd addr

$$50 \begin{cases} 1000, & 1001 \\ 1002, & 1003 \\ & \vdots \\ 1098, & 1099 \end{cases}$$

51 houses ⇒ two must have #s from the same group.

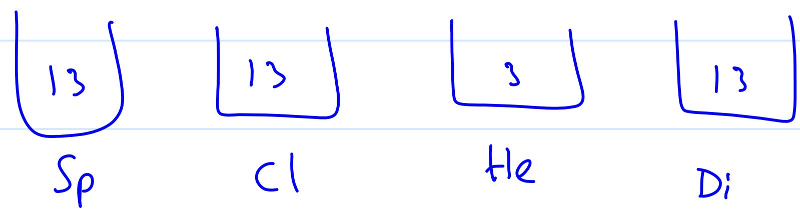
Example 7, Section 6.2. Consider a standard deck of 52 cards.

(a) How many cards must be drawn to ensure that at least three of these cards are from the same suit.

2	2	2	2
Spades	Clubs.	Hearts	Diamonds

$4(3-1)+1 = 9$

(b) How many cards must be drawn at random to guarantee that at least three hearts are selected?



$39 + 3 = 42$

46. Let $n_1, n_2, \ldots, n_t$ be positive integers. S.T. if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed in t boxes, then for some i, $i = 1, \ldots, t$, the ith box contains n_i objects.

Assume box i, $i = 1, \ldots, t$, has fewer than n_i objects.

$\Rightarrow$ box i has at most $n_i - 1$ objects.

box 1	$\leq$	$n_1 - 1$
box 2	$\leq$	$n_2 - 1$
	$\vdots$	
box t	$\leq$	$n_t - 1$

The maximum number of objects that can be placed in the boxes w/o contradicting the assumption

$$= (n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1)$$

$$= n_1 + n_2 + \cdots + n_t - t$$

Trying to place one more object, the $(n_1 + \cdots + n_t - t + 1)^{th}$ object, will make one of the boxes, say i, to have $n_i - 1 + 1 = n_i$ objects.