Permutations 3/5/12

Note Title

A permutation of a set of distinct elements is an ordered sequence/arrangement of these elements.

Example: Three students A, B, C stand in line for a picture taking.

# of arrangements:

A B C
A C B
B A C
B C A
C A B
C B A

6 different arrangements

3 choices 2 1

By product rule,
$3 * 2 * 1 = 6$ possibilities

This is a 3-permutation of 3 students.

An ordered arrangement of r elements out of n distinct elements is called an r-permutation.

Note: $0 \le r \le n$ and $n \geq 0$.

Notation: $P(n, r)$

For the above problem, we counted 3-permutations of 3 distinct elements:

$$P(3,3) = 3 * 2 * 1 = 3! = 6$$

Example: If the number of available students is 5 and the line should have three students, then the number of ways to form a line is

$$P(5,3) = 5 * 4 * 3$$

5 choices 4 3

Theorem 1. $P(n,r) = n(n-1)\cdots(n-r+1)$, $1 \le r \le n$, $n > 0$

Proof: use the product rule

$$\begin{array}{cccccc} \text{choices} & n & n-1 & n-2 & & (n-r+1) \\ & \downarrow & \downarrow & \downarrow & & \downarrow \\ & \_ & \_ & \_ & \cdots & \_ \\ & 1 & 2 & 3 & & r \end{array}$$

$\therefore P(n,r) = n\cdot(n-1)\cdot\ \cdots\ (n-r+1)$

$P(n,0) \triangleq 1$ and $P(n,n) = n!$

defined

Corollary: $P(n,r) = \dfrac{n!}{(n-r)!}$

$$\text{RHS} = \frac{n!}{(n-r)!} = \frac{n\cdot(n-1)\cdots(n-r+1)\cancel{(n-r)}\cdots\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{2}\cdot\cancel{1}}$$

right hand side

$$= n(n-1)\cdots(n-r+1)$$
$$= P(n,r)$$

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5\cdot4\cdot3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}} = 5\cdot4\cdot3$$

Example: How many ways are there to select three -- first, second, third place -- winners out of 100?

$P(100,3) = 100\cdot 99\cdot 98$

Problem 2. S={a,b,c,d,e,f,g}. |S|=7.

# of permutations of S is $P(7,7) = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 7!$

2'. # of 3-permutations of S is $P(7,3) = 7 \times 6 \times 5 = 210$

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{5040}{24} = 210$$

5a $P(6,3) = 6 \times 5 \times 4 = 120$

Example 7. How many ways can the letters A, B, C, D, E, F, G, H be permuted to form 8-letter strings that contain the substring ABC.

Treat ABC as one super letter X.

Now permute X, D, E, F, G, H $\Rightarrow P(6,6) = 6! = 720$

Alternate solution:

permute D E F G H $\quad P(5,5) = 120 = 5!$

$\uparrow$ D $\uparrow$ E $\uparrow$ F $\uparrow$ G $\uparrow$ H $\uparrow$

'ABC' can go into one of these places

$P(5,5) \cdot 6 = 5! * 6 = 6! = 720$

Combinations

Example: How many different committees of three students can be formed from a group of 4?

A, B, C, D

Note: The order of selection does not matter.

\# of ways to leave out one of four is 4.

$\therefore$ # of ways to select a committee of 3 out of 4 is 4.

Suppose the order of selection matters:

$\therefore$ If order matters then # ways to form the "ordered" committee is $P(4,3) = 4 \times 3 \times 2 = 24$

4 choices ↓ ___ 3 ↓ ___ 2 ↓ ___

Committee positions

Suppose A, B, C are selected. → 1 of 4 possible unordered choices.

If order matters, then with the same three you can have

$$\left.\begin{matrix} ABC \\ ACB \\ BAC \\ BCA \\ CAB \\ CBA \end{matrix}\right\} 6 \text{ possibilities} = 3! = P(3,3)$$

$$\therefore \text{ \# unordered choices} = \frac{\text{\# ordered choices}}{P(3,3)} = \frac{24}{6} = 4$$

An r-combination of the elements of a set is an unordered selection of r elements from the Set.

Notation: $C(n,r)$ or $\binom{n}{r}$ – an r-combination of n elements.

also called binomial coefficient.

Theorem 2 $$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$\binom{n}{r}$ is an r-comb. of n elements.

Each r-comb. gives $P(r,r)$ distinct orders.

$$\underbrace{\binom{n}{r}}_{\#\ r\text{-comb.}} \cdot \underbrace{P(r,r)}_{\#\ \text{orders for each comb.}} = \underbrace{P(n,r)}_{\#\ \text{ordered selections}}$$

$$\boxed{\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{(n-r)!\,r!}}$$

Example 11. The number of 5-card poker hands from a standard deck of 52 cards is

$$\binom{52}{5} = \frac{52!}{(52-5)!\ 5!} = \frac{52!}{47!\ 5!}$$

The number of 47-card poker hands from a standard deck of cards is

$$\binom{52}{47} = \frac{52!}{(52-47)!\ 47!} = \frac{52!}{5!\ 47!}$$

Corollary 2

$$\binom{n}{r} = \binom{n}{n-r}, \qquad r, n \text{ nonnegative. ints.}$$

Algebraic manipulation

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \qquad (\text{by def. \& theorem 2})$$

$$\binom{n}{n-r} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Combinatorial proof:
Counting arguments are used to show that both sides of an identity count the same objects in different ways.

$\binom{n}{r}$ = give n objects, select/choose r

It is the same as picking $(n-r)$ objects that should be discarded so that the remaining r objects are selected.
This is counted as $\binom{n}{n-r}$.

Since they both count the same $\binom{n}{r} = \binom{n}{n-r}$

Problems from Section 5.3

Problem 23. How many ways are there to place eight men and five women in a line so that no two women stand next to each other?

- place eight men — $8! = P(8,8)$

$\square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square\ \underline{M}\ \square$

- place five women in the 9 spots among men — $P(9,5)$

$\therefore$ Total # of ways $= 8! * P(9,5)$

Problem 33. A department has 10 men and 15 women. How many ways are there to form a committee of six
(a) if there must be equal number of men and women?

$$\underbrace{\binom{10}{3}}_{\text{ways to select 3 men}} \times \underbrace{\binom{15}{3}}_{\text{ways to select 3 women}}$$

$$= \frac{10 \times 9 \times 8}{3!} \times \frac{15 \times 14 \times 13}{3!} = (5 \times 3 \times 8) \times (5 \times 7 \times 13)$$

$$= 120 \times 455$$

(b) if there must be more women than men

Women	men	
$\rightarrow$ 4	2	$\Rightarrow \binom{15}{4} \cdot$
$\rightarrow$ 5	1	$\Rightarrow \binom{15}{5} \cdot \binom{10}{1}$
$\rightarrow$ 6	0	$\Rightarrow \binom{15}{6} \cdot \binom{10}{0}$

Total # of ways $= \binom{15}{4} \cdot \binom{10}{2} + \binom{15}{5} \cdot \binom{10}{1} + \binom{15}{6} \cdot \binom{10}{0}$

35. Count the number of bit strings that contain exactly eight 0s and ten 1s if every 0 must be followed by a 1?

Since each 0 must be followed by a 1,
create a '01' token.

Need to have eight 0s $\Rightarrow$ eight tokens $\Rightarrow$ 16 bits
Since the total string length is 18,
the other two bits must be 1s.

$\therefore$ The problem is choosing the positions of
eight tokens from $8+2=10$ spaces.

$$\binom{10}{8} = \binom{10}{2} = \frac{10 \times 9}{2!} = 45$$

Problem.
Ten distinct paintings are to be allocated to k office rooms so that (i) no room gets more than one painting and (ii) there are either no paintings left or no empty rooms left. Find the number of ways of doing this if (a) k=14 and (b) k=6.

(a) $k = 14$ rooms, 10 paintings $P(14, 10)$

(b) $k = 6$ $P(10, 6)$

Problem.
Redo the above problem with ten identical posters instead of paintings.

$k = 14$ rooms $C(14, 10) = \binom{14}{10}$

$k = 6$ rooms $\binom{6}{6}$

Problems not solved in class

Problem 21. Give the number of permutations of the seven letters A, B, C, D, E and F

(a) containing the string 'BCD'.

Create a super letter 'BCD'. Now permute A, 'BCD', E, F and G in 5! ways.

(c) strings have 'BA' and 'GF'.

$\therefore$ Permutation of 'BA', C, D, E, 'GF' $\Rightarrow 5!$

(f) strings that contain 'BCA' and 'ABF'? 0; cannot have both BCA & ABF.

Problem 11. Count the number of 10-bit strings with the following properties.

(a) # strings with exactly four 1s $= \binom{10}{4} = \frac{10!}{6!\,4!}$

$$\binom{10}{4} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$$

(b) at most four 1s

Can have zero 1s, one 1, two 1s, three 1s or four 1s

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4}$$

$$= \frac{\cancel{10!}}{\cancel{10!}\,\cancel{0!}}_{1} + \frac{10}{1} + \frac{10\times 9}{2\times 1} + \frac{10\times 9\times 8}{3\times 2\times 1} + 210$$

$$= \underbrace{1 + 10 + 45 + 120}_{176 \text{ = \# strings with at most three 1s}} + 210$$

$$= 386$$

(c) At least four 1s.

$$\binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}$$

$$= \sum_{k=4}^{10} \binom{10}{k}$$

Total # of Strings $= 2^{10} = 1024$

Required count $= 1024 - 176$

(d) equal # of 1s and 0s $= C(10,5) = \binom{10}{5} = \frac{10!}{5!\,5!}$

13. Count the number of ways n men and n women can form a line if men and women must alternate.

2 ways

$$\begin{array}{cccccc} M & W & M & W & \cdots & M & W \\ \_ & \_ & \_ & \_ & \cdots & \_ & \_ \\ 1 & 2 & 3 & 4 & & 2n-1 & 2n \end{array}$$

$$\begin{array}{cccccc} W & M & W & M & \cdots & M \\ \_ & \_ & \_ & \_ & \cdots & \_ \\ 1 & 2 & 3 & 4 & & 2n \end{array}$$

ways to fill n spots with n men $= P(n,n) = n!$

ways to fill n spots with n women $= P(n,n) = n!$

The first person can be a man or a woman $= 2$

$\therefore$ Total # ways $= 2 * (n!) * (n!) = 2(n!)^2$

17. A set has 100 elements. What is the number of subsets of this set with more than two elements?

Total # of subsets $= 2^{100}$ ← includes subsets with

0 elements $= C(100,0) = 1$

1 element $= C(100,1) = 100$

2 elements $= C(100,2) = \frac{100 \times 99}{2} = 49500$

Required count $= 2^{100} - C(100,0) - C(100,1) - C(100,2)$

$= 2^{100} - 1 - 100 - 49500 =$

19. A coin is flipped 10 times. each flip results in a head or a tail. Calculate the number of possible outcomes for the following cases.

(a) total outcomes $= 2^{10} = 1024$

Choices

$\underset{1}{\underline{2}}\ \underset{2}{\underline{2}}\ \cdots\ \underset{10}{\underline{2}} = 2^{10}$

(b) Contain exactly two heads

$= C(10,2) = \binom{10}{2} = \frac{10 \times 9}{2} = 45$

(c) at most three tails : $\underset{\uparrow \text{ 0 tails}}{C(10,0)} + \underset{\text{1 tail}}{C(10,1)} + \underset{\uparrow \text{ 2 tails}}{C(10,2)} + \underset{\uparrow \text{ 3 tails}}{C(10,3)}$

$= 176$

(d) Same # of tails & heads $\Rightarrow$ 5 tails and 5 heads

$\binom{10}{5} = 252$

20. How many bit strings of length 10 have

(a) exactly three 0s? — similiar to 19 (b)

(b) more 0s than 1s?

↳ need to count all strings with six 0's

seven 0's

eight 0's

nine 0's

ten 0's

Use sum rule to add them up.

(c) at least seven 1s.

↳ all strings that have exactly seven 1s

or eight 1s

or nine 1s

or ten 1s

(d) at least three 1s?

25. 100 tickets 1, ..., 100 are sold to 100 different people for a drawing in which four different prizes are awarded including a grand prize. Hou many ways are these to award prizes if

(a) There are no restrictions? 100^4

$$\underset{\_\_}{\overset{100}{\downarrow}} \quad \underset{\_\_}{\overset{100}{\downarrow}} \quad \underset{\_\_}{\overset{100}{\downarrow}} \quad \underset{\_\_}{\overset{100}{\downarrow}}$$

any person can win at most one prize?

$$100 \cdot 99 \cdot 98 \cdot 97$$

$$P(100,4) = \frac{100!}{96!} =$$

$$\underset{\_\_}{\overset{100}{\downarrow}} \quad \underset{\_\_}{\overset{99}{\downarrow}} \quad \underset{\_\_}{\overset{98}{\downarrow}} \quad \underset{\_\_}{\overset{97}{\downarrow}}$$

(b) The person holding ticket 47 wins the grand prize?

$$\underset{GP}{\overset{1}{\downarrow}} \quad \underset{\_\_}{\overset{99}{\downarrow}} \quad \underset{\_\_}{\overset{98}{\downarrow}} \quad \underset{\_\_}{\overset{97}{\downarrow}} \qquad 1 \cdot 99 \cdot 98 \cdot 97 \qquad P(99,3)$$

(c) The person holding ticket 47 wins one of the prizes?

give one of the prizes to 47 = 4

Give 3 prizes to 3 out of 99 $\quad \boxed{P(99,3)} \quad C(99,3)$

$\therefore$ Total $= 4 \cdot P(99,3)$

$$P(99,3) \Rightarrow \frac{99!}{96!} \qquad C(99,3) \to \frac{99!}{96!\,3!}$$

(d) The person holding ticket # 47 does not win a prize? P(99,4)

(i) The grand prize winner is a person holding ticket 19,47,73, or 97? 4* P(99,3)