Binomial Coefficients 3/7/12

Count all 4-bit strings

$2^4 = 16$

Choices

$\underline{2}\quad \underline{2}\quad \underline{2}\quad \underline{2}$

$2 \times 2 \times 2 \times 2 = 2^4$

$\boxed{0000}$	✓1000
✓0001	× 1001
✓0010	× 1010
× 0011	– 1011
✓0100	× 1100
× 0101	– 1101
× 0110	1110
– 0111	1111

n_i = # of 4-bit strings with 'i' 1s.

$n_0 + n_1 + n_2 + n_3 + n_4$

= all 4-bit strings $= 2^4$

n_0 = # of 4-bit strings with zero 1s $= \binom{4}{0} = 1$

$n_1 = \binom{4}{1} = 4$

$n_2 = \binom{4}{2} = \frac{4 \times 3}{2!} = \frac{4 \times 3}{2 \times 1} = 6$

$n_3 = \binom{4}{3} = \frac{4 \times 3 \times 2}{3!} = 4 = \binom{4}{1}$

$\binom{n}{r} = \binom{n}{n-r}$

$n_4 = \binom{4}{4} = 1$

$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4$

1 4 6 4 1

Binomial coefficients

Binomial Theorem

Binomial expression – a sum of two terms, e.g. $x+y$

Powers of binomial expressions – $(x+y)^4$, $(x+y)^3$, $(a-b)^{10}$, ...

Binomial coefficients are $\binom{n}{r}$, $n, r \geq 0$, $r \leq n$

Expansion of powers of binomial exp.

$$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0} \cdot x^2 + \binom{2}{1} \cdot xy + \binom{2}{2} y^2$$

$$(x+y)(x+y) = \binom{2}{0} x^2 \cdot y^0 + \binom{2}{1} \cdot x^1 y^1 + \binom{2}{2} \cdot x^0 y^2$$

pick x & x $\quad x^2 \cdot y^0 = \binom{2}{0}$ ways

pick x & y $\quad xy$

pick y & x $\quad yx = xy$ $\quad \binom{2}{1}$

pick y & y $\quad y^2 \cdot x^0$ – $\binom{2}{2}$ ways

$$\binom{n}{r} = \binom{n}{n-r}$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$\begin{aligned}(x+y)\cdot(x+y)\cdot(x+y) &= \binom{3}{0} x^3y^0 \\ &+ \binom{3}{1} x^2y^1 \\ &+ \binom{3}{2} xy^2 \\ &+ \binom{3}{3} x^0y^3\end{aligned}
\equiv
\begin{aligned}&\binom{3}{3} x^3 y^0 \\ &+ \binom{3}{2} x^2y^1 \\ &+ \binom{3}{1} xy^2 \\ &+ \binom{3}{0} x^0y^3\end{aligned}$$

Binomial Theorem

$$(x+y)^n = \underline{\binom{n}{0}} x^n + \underline{\binom{n}{1}} x^{n-1} y^1 + \underline{\binom{n}{2}} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k, \quad n \geq 0$$

Combinatorial proof:

$$(x+y)^n = \underline{(x+y)\cdot(x+y)\cdot \cdots (x+y)} \quad (n \text{ times})$$

Take a term such as $x^{n-j} \cdot y^j$, $0 \leq j \leq n$

$\binom{n}{j}$ is the number of ways 'j' y terms are picked from n binomial expressions.

and x is picked from the remaining expressions.

$\therefore$ The coefficient of $x^{n-j} \cdot y^j$ is $\binom{n}{j}$.

Example

$$(x+y)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$x=1,\ y=1$ $\qquad LHS = (1+1)^4 = 2^4 = 16$

$$RHS = 1^4 + 4\cdot 1^3 \cdot 1 + 6 \cdot 1^2 \cdot 1^2 + 4 \cdot 1 \cdot 1^3 + 1^4$$

$$= 1+4+6+4+1 = 16$$

Examples

Coefficient of $x^{12} \cdot y^{13}$ in $(x+y)^{25}$? $\binom{25}{12}$, or $\binom{25}{13}$

Coefficient of $x^{12} \cdot y^{13}$ in $(2x-3y)^{25}$?

The term that contains $x^{12} \cdot y^{13}$ is

$$\binom{25}{12} \cdot (2x)^{12} \cdot (-3y)^{13} = \boxed{-\binom{25}{12} \cdot 2^{12} \cdot 3^{13}} \cdot x^{12} \cdot y^{13}$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n} y^n, \quad n \geq 0 \quad \text{—} (1)$$

put $x = y = 1$

$$(1+1)^n = \binom{n}{0} \cdot 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot 1 + \cdots + \binom{n}{n} \cdot 1^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

Put $x = 1,\ y = -1$ in (1)

$$(1-1)^n = \binom{n}{0} 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot (-1) + \binom{n}{2} \cdot 1^{n-2} \cdot (-1)^2 + \cdots + \binom{n}{n} (-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}$$

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$$

Pascal's Triangle

$n=0$ $\qquad \binom{0}{0} \triangleq 1$

$n=1$ $\qquad \binom{1}{0}=1 \quad \binom{1}{1}=1$

$n=2$ $\qquad \binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$

$n=3$ $\qquad \binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1$

$n=4$ $\qquad \binom{4}{0}=1 \quad \binom{4}{1}=4 \quad \binom{4}{2}=6 \quad \binom{4}{3}=4 \quad \binom{4}{4}=1$

Pascal's Identity $\qquad \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Combinatorial argument:

Consider a set of $n+1$ elements: $\{a, x_1, x_2, \ldots, x_n\}$

$\#$ ways to choose k elements $1 \le k < n$, is $\binom{n+1}{k}$

Another approach to count the same

have 'a' in the k elements chosen

Do not have the 'a' in the k elements chosen

By Sum rule $\qquad \binom{n}{k-1} + \binom{n}{k}$ = all combinations of k elements

$= \binom{n+1}{k}$

Algebraic proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!\,(k-1)!} + \frac{n!}{(n-k)!\;k!}$$

$$= \frac{k\,n! + (n-k+1)\,n!}{(n-k+1)!\;k!}$$

$$= \frac{\cancel{k\cdot n!} + \overbrace{(n+1)\cdot n!} - \cancel{k\cdot n!}}{(n-k+1)!\;k!}$$

$$= \frac{(n+1)!}{(n+1-k)!\;k!} = \binom{n+1}{k} \quad \text{by def.}$$

Revisit the password counting problem (Section 6.1)

P6: 6-char (letter or digit) string with at least one digit. $= 36^6 - 26^6$

Passwords with exactly 1 digit $= \binom{6}{1} \cdot 10^1 \cdot 26^5$

2 digits $= \binom{6}{2} \cdot 10^2 \cdot 26^4$

3 $= \binom{6}{3}\, 10^3\, 26^3$

4 $= \binom{6}{4}\, 10^4\, 26^2$

5 $= \binom{6}{5}\, 10^5\, 26^1$

6 $= \binom{6}{6}\, 10^6\, 26^0$

Valid

PW with 0 digits $= \binom{6}{0}\, 10^0\, 26^6$ Invalid

$$(26+10)^6 = \sum_{k=0}^{6} \binom{6}{k} \cdot (26)^{6-k} \cdot (10)^k$$

$$= \binom{6}{0} \cdot 26^6 + \underbrace{\binom{6}{1} 26^5 \cdot 10^1 + \binom{6}{2} \cdot 26^4 \cdot 10^2 + \binom{6}{3} \cdot 26^3 \cdot 10^3 + \binom{6}{4} \cdot 26^2 \cdot 10^4 + \binom{6}{5} \cdot 26^1\, 10^5 + \binom{6}{6}\, 10^6}_{\text{valid}}$$

$$36^6 - 26^6 = \underbrace{\binom{6}{1} 26^5 \cdot 10^1 + \binom{6}{2} \cdot 26^4 \cdot 10^2 + \binom{6}{3} \cdot 26^3 \cdot 10^3 + \binom{6}{4} \cdot 26^2 \cdot 10^4 + \binom{6}{5} \cdot 26^1\, 10^5 + \binom{6}{6}\, 10^6}_{\text{valid}}$$

Vandermondis Identity (VI)

Theorem 3

$$\binom{m+n}{r} = \sum_{k=0}^{r}\left[\binom{m}{r-k}\binom{n}{k}\right] \qquad 0 \le r \le m,\quad 0 \le r \le n$$

Proof: Combinatorial Proof:

Select a committee of r people from m men and n women.

$\binom{m+n}{r}$ – Straight forward appl of combinations

Committee with 0 men   with 1 man   2 men   ...   r men

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} \cdots + \binom{m}{r}\cdot\binom{n}{0}$$

Cor:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

put $m = n = n$ and $r = n$ in V.I.

Problems on Binomial Coefficients, Section 5.4

21. $1 \le k \le n$. S.T. $k \binom{n}{k} = n \cdot \binom{n-1}{k-1}$

Consider the selection of a committee of k people with a chairperson out of n people.

LHS: Choose k-people for the Committee $\binom{n}{k}$

choose a chair from the committee members $\binom{k}{1}$

$\therefore$ By Product Rule, $\binom{k}{1} \cdot \binom{n}{k}$

RHS: Choose the chair person first $\binom{n}{1}$

Choose the rest of the comm. $\binom{n-1}{k-1}$

$\therefore$ By Product Rule, $\binom{n}{1} \cdot \binom{n-1}{k-1}$