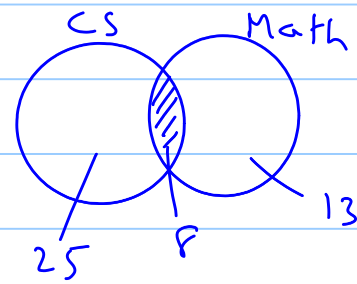
Principle of Inclusion-Exclusion (PIE) 3/26/12

Note Title

1. Disc. math class - each Student is a math or CS major
There are 25 CS majors, 13 math majors, 8 both
CS & math majors
what is the total # of Students



$25 + 13 - 8 = 30$

$|A| + |B| - |A \cap B| = |A \cup B|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

P_1 = a Student is a CS major $\quad N(P_1)$ = # of CS majors

P_2 = a Student is a math major $\quad N(P_2)$ = # of math majors

$N(P_1 P_2)$ = # of Students with P_1 & P_2 satisfied
= # of Students, each majoring in both CS & Math

$\therefore$ Total # of Students $= N(P_1) + N(P_2) - N(P_1 P_2)$
$= 25 + 13 - 8$

2. $\overbrace{\#\text{ students taking Spanish}}^{P_1} = 1232$ $\quad N(P_1) = 1232$

$\#$ students $\overbrace{\text{taking French}}^{P_2} = 879$ $\quad N(P_2) = 879$

$\overbrace{\text{Russian}}^{P_3} = 114$ $\quad N(P_3) = 114$

taking both S & F $= 103$ $\quad N(P_1P_2) = 103$

S & R $= 23$ $\quad N(P_1P_3) = 23$

F & R $= 14$ $\quad N(P_2P_3) = 14$

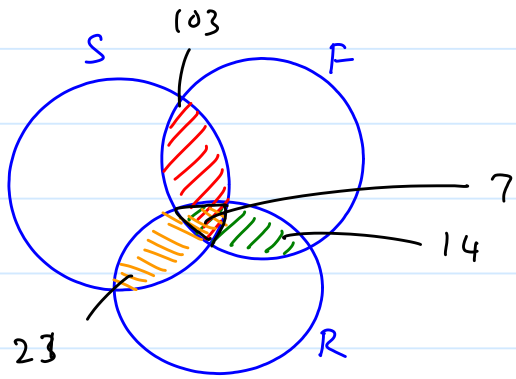
taking S, F & R $= 7$ $\quad N(P_1P_2P_3) = 7$

$N(P_1 \vee P_2 \vee P_3)$ = the total # of students taking S, F or R

$$= N(P_1) + N(P_2) + N(P_3) - N(P_1P_2) - N(P_1P_3) - N(P_2P_3) + N(P_1P_2P_3)$$

$$= 1232 + 879 + 114 - 103 - 23 - 14 + 7$$

$$= 2092$$



If there are $N = 3000$ students in the school, how many students are not taking any of the S, F, or R courses?

$$N(P_1'P_2'P_3') = N - N(P_1 \vee P_2 \vee P_3)$$

$$3000 - 2092 = 908$$

Generalized Principle of Inclusion and Exclusion (PIE)

n sets - $A_1, A_2, \ldots, A_n$ $\quad |A_i| =$ Size of A_i

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| \longleftarrow \binom{n}{1} \text{ terms}$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \longleftarrow \binom{n}{2} \text{ terms}$$

$$+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \quad \binom{n}{3} \text{ terms}$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4| - \cdots - |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n| \longleftarrow \binom{n}{4} \text{ terms}$$

$$\cdots \qquad \vdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \longleftarrow \binom{n}{n} \text{ terms}$$

A more compact notation

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$- \cdots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Applications of PIE

$P_1, \ldots, P_n$ are n properties of N elements

Let A_i = Subset of these N elements, with property P_i

$N(P_i) = |A_i|$

$N(P_i P_j P_k) = |A_i \cap A_j \cap A_k|$

$$N(P_1 \vee P_2 \vee \cdots \vee P_n) =$$

$$\sum_{1 \le i \le n} N(P_i) - \sum_{1 \le i < j \le n} N(P_i \cap P_j) + \sum_{1 \le i < j \le n} N(P_i P_j P_k) - \cdots + (-1)^{n+1} N(P_1 P_2 \cdots P_n)$$

is the total number of elements that satisfy one or more of $P_1, P_2, \ldots, P_n$.

$$N(P_1' P_2' \cdots P_n') = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i \cap P_j)$$

$$- \cdots$$

$$+ (-1)^n N(P_1 P_2 \cdots P_n)$$

Example, Section 8.6, Page 560

What is the number of primes between 2 and 100.

$\lfloor \sqrt{100} \rfloor = 10$ 2, 3, 5, 7 are ≤ 10.

P_2 = property that a number is divisible by 2

P_3 = " 3

P_5 = " 5

P_7 = " 7

$\therefore$ Total # of primes $N(P_2' P_3' P_5' P_7') + 4$

$\binom{4}{1}$

$N(P_2) = \lfloor \frac{100}{2} \rfloor = 50$ $N(P_5) = \lfloor \frac{100}{5} \rfloor = 20$

$N(P_3) = \lfloor \frac{100}{3} \rfloor = 33$ $N(P_7) = \lfloor \frac{100}{7} \rfloor = 14$

$\binom{4}{2}$

$N(P_2 P_3) = \lfloor \frac{100}{LCM(2,3)} \rfloor = \lfloor \frac{100}{6} \rfloor = 16$ $N(P_2 P_5) = \lfloor \frac{100}{10} \rfloor = 10$

$N(P_2 P_7) = \lfloor \frac{100}{14} \rfloor = 7$ $N(P_3 P_5) = \lfloor \frac{100}{3 \cdot 5} \rfloor = 6$ $N(P_3 P_7) = \lfloor \frac{100}{21} \rfloor = 4$

$N(P_5 P_7) = \lfloor \frac{100}{35} \rfloor = 2$

$\binom{4}{3}$

$N(P_2 P_3 P_5) = \lfloor \frac{100}{30} \rfloor = 3$ $N(P_2 P_3 P_7) = \lfloor \frac{100}{42} \rfloor = 2$

$N(P_2 P_5 P_7) = \lfloor \frac{100}{70} \rfloor = 1$ $N(P_3 P_5 P_7) = \lfloor \frac{100}{105} \rfloor = 0$

$\binom{4}{4}$

$N(P_2 P_3 P_5 P_7) = \lfloor \frac{100}{2 \cdot 3 \cdot 5 \cdot 7} \rfloor = 0$

# of numbers that are multiples of 2, 3, 5, or 7

$= N(P_2) + N(P_3) + N(P_5) + N(P_7)$

$- N(P_2P_3) - N(P_2P_5) - N(P_2P_7) - N(P_3P_5) - N(P_3P_7) - N(P_5P_7)$

$+ N(P_2P_3P_5) + N(P_2P_3P_7) + N(P_2P_5P_7) + N(P_3P_5P_7)$

$- N(P_2P_3P_5P_7)$

$= 50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2$

$+ 3 + 2 + 1 + 0 - 0$

$= 78$

$N(P_2' P_3' P_5' P_7') = 99 - 78 = 21$

$\therefore$ Total primes between 2 and 100 $= 21 + 4 = 25$

Examplel 1, Section 8.6

Find the # of solutions to $x_1+x_2+x_3=11$, where

$0 \le x_1 \le 3$, $0 \le x_2 \le 4$, $0 \le x_3 \le 6$, x_1, x_2, x_3 are integers.

W/o Constraints on x_i: # possible solutions $\binom{3-1+11}{11} = \binom{13}{11} = \binom{13}{2} = \frac{13 \cdot 12}{2}$

$= 78$

P_1 = # solutions with $x_1 \geq 4$ $\quad N(P_1) = \binom{3-1+7}{7} = \binom{9}{2} = \frac{9 \cdot 8}{2} = 36$

$\underset{x_1}{\boxed{4}}$ $\underset{x_2}{\boxed{\ }}$ $\underset{x_3}{\boxed{\ }}$ 7 left $\quad \binom{9}{7} = \binom{9}{9-7} = \binom{9}{2}$

P_2 = # solutions with $x_2 \geq 5$ $\quad N(P_2) = \binom{3-1+6}{6} = \binom{8}{2} = \frac{8 \cdot 7}{2} = 28$

$\underset{x_1}{\boxed{\ }}$ $\underset{x_2}{\boxed{5}}$ $\underset{x_3}{\boxed{\ }}$ 6 left

P_3 = # solutions with $x_3 \geq 7$ $\quad N(P_3) = \binom{3-1+4}{4} = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15$

$\underset{x_1}{\boxed{\ }}$ $\underset{x_2}{\boxed{\ }}$ $\underset{x_3}{\boxed{7}}$ 4 choices

$N(P_1 P_2) = \binom{3-1+2}{2} = \binom{4}{2} = 6$ $\quad x_1 \geq 4, x_2 \geq 5$ $\quad \underset{x_1}{\boxed{4}}$ $\underset{x_2}{\boxed{5}}$ $\underset{x_3}{\boxed{\ }}$ 2 choices left

$N(P_1 P_3) = \binom{3-1+0}{0} = 1$ $\quad x_1 \geq 4, x_3 \geq 7$ $\quad \underset{x_1}{\boxed{4}}$ $\underset{x_2}{\boxed{\ }}$ $\underset{x_3}{\boxed{7}}$ 0 choices left

$N(P_2 P_3) = 0$ $\quad x_2 \geq 5, x_3 \geq 7$ $\quad \underset{x_1}{\boxed{\ }}$ $\underset{x_2}{\boxed{5}}$ $\underset{x_3}{\boxed{7}}$ X Not feasible

$N(P_1 P_2 P_3) = 0$ $\quad x_1 \geq 4, x_2 \geq 5, x_3 \geq 7$

$N(P_1 \vee P_2 \vee P_3) =$ # of Solutions with P_1 or P_2 or P_3

$= N(P_1) + N(P_2) + N(P_3) - N(P_1P_2) - N(P_1P_3) - N(P_2P_3) + N(P_1P_2P_3)$

$= 36 + 28 + 15 - 6 - 1 - 0 + 0 = 72$

# of Desired solutions $N(P_1'P_2'P_3') = N - N(P_1 \vee P_2 \vee P_3) = 78 - 72 = 6$