Probability

4/4/12

Note Title

Random Experiment (RE) –
an experiment whose outcome is not known in advance, but the set of all possible outcomes is known.

Sample point – is an outcome of a RE.

Sample space, Ω = {all possible outcomes of a RE}

$\Omega = \{e : e \text{ is an outcome of RE}\}$

Examples 1. A coin flip is a RE.

$\Omega = \{\text{Head}, \text{Tail}\}$

sample point   sample point

2. Roll a six-faced die – RE

$\Omega = \{1, 2, 3, 4, 5, 6\}$

3. Flip a coin until head occurs.

H – head   T – tail

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

required 1 flip   2 flips   3 flips   4 flips ...

Event: a subset of the sample space Ω, satisfying certain axioms.

Example 4 Roll a die. $\Omega = \{1, 2, 3, 4, 5, 6\}$

Let A = event that the die roll results in a prime number

$A = \{2, 3, 5\}$ [Note: 1 is not a prime #

$A \subseteq \Omega$

B = event that a six occurs $= \{6\} \subseteq \Omega$

C = event that a seven occurs $= \{\ \} = \emptyset \subseteq \Omega$

impossible event null set

D = event that an even # occurs $= \{2, 4, 6\} \subseteq \Omega$

$A \cap D$ = event that an even #'ed prime number occurs $= \{2\}$

A' or $\bar{A}$ = a prime number does not occur $= \{1, 4, 6\}$

$\bar{A} = \Omega - A = \{1, 2, 3, 4, 5, 6\} - \{2, 3, 5\}$

Complementary event of A.

A and $\bar{A}$ partition the sample space

(1) A and $\bar{A}$ are not empty

(2) $A \cap \bar{A} = \emptyset$

(3) $A \cup \bar{A} = \Omega$

Example 5.

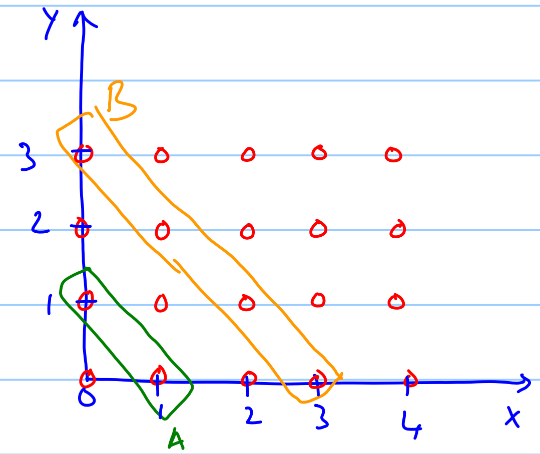
A system consists of two subsystems: one with four components and the other with three components. We are interested in only working components of the system. Components fail randomly.

The event space and events on the working condition of the system can be formulated as follows.

$$\Omega = \{(x, y) : 0 \le x \le 4,\ 0 \le y \le 3\}$$

\# working in SS1   \# working in SS2

$$= \{(0,0), (0,1), (0,2), (0,3), (1,0), \ldots, (4,3)\}$$



A = exactly one component is working

$$= \{(0,1), (1,0)\} = \{(x,y) : x+y=1\}$$

B = exactly three components are working

$$= \{(0,3), (1,2), (2,1), (3,0)\} = \{(x,y) : x+y=3\}$$

C = none of the SS2 components are working

$$= \{(x,0) : 0 \le x \le 4\}$$

Probability measure

A probability measure assigns a numerical value in [0, 1] for the probability or chance that an event occurs.

A probability of 0 => the event never occurs => an impossible event

A probability of 1 => the event always occurs => a certain event

Other events have probabilities between 0 and 1.

Classical probability

-- Ω is finite
-- all outcomes are equally likely

$$\therefore P(A) = \text{Prob. that event } A \text{ occurs} = \frac{|A|}{|\Omega|}$$

Example: Roll a fair die. $\Omega = \{1, 2, 3, 4, 5, 6\}$, $|\Omega| = 6$

each sample point can occur with prob. $\frac{1}{6}$

A = a prime # occurs = $\{2, 3, 5\}$ $\quad P(A) = \frac{3}{6} = \frac{1}{2}$

C = a seven occurs = $\{\ \}$ or ϕ $\quad P(C) = 0$

Formal definition of probability measure

Let $\mathbb{F}$ = family of events of Ω

$\Omega \in \mathbb{F}$, ϕ = impossible event $\in \mathbb{F}$

Probability measure P is a real-valued function on $\mathbb{F}$.

$$P : \mathbb{F} \longrightarrow \overbrace{[0, 1]}^{\{p :\, 0 \le p \le 1\}}$$

Axioms of probability measure

1. $P(A) \geq 0$ for every event A
2. $P(\Omega) = 1$
3. $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive.

Example : An urn contains four blue balls and five red balls. What is the probability that a blue ball is chosen if a ball is taken out of the urn at random.

A = a blue ball is chosen

$|A| = 4 \qquad |\Omega| = 9$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{4}{9}$$

Ex. 3 Lottery: pick four digits from 0, ..., 9.
repetition allowed, order matters.

A - Grand prize - if all four digits are picked correctly

B - Small prize - if three of the four digits are picked correctly.

$\Omega = \{\text{all 4-digit \#s}\} = \{0000, 0001, \ldots, 9999\}$

$|\Omega| = 10^4$ $\qquad \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}$

$|A| = 1$

$P(A) = \frac{1}{10^4} = 0.0001$

1 2 3 4

_ 2 3 4

$|B| = \binom{4}{1}\cdot\binom{9}{1} = 36$

\# ways to place the incorrect digit

\# incorrect digit choices

$P(B) = \frac{36}{10^4} = 0.0036$

Example 7 [Rosen, Section 7.1]

A bin contains 50 balls labeled 1, 2, ..., 50. What is the probability that balls labeled 11, 4, 17, 39 and 23 are drawn in five consecutive random draws if

(a) balls are replaced (put back in the bin) after each draw?

(b) balls are not replaced after they are drawn?

A = draw balls 11, 4, 17, 39 and 23 in five consecutive draws.

choices

$\underline{50}$ $\underline{50}$ $\underline{50}$ $\underline{50}$ $\underline{50}$

(a) $|\Omega| = 50^5$

$|A| = 1$

$P(A) = \dfrac{1}{50^5}$.

choices

$\underline{50}$ $\underline{49}$ $\underline{48}$ $\underline{47}$ $\underline{46}$

(b) $|\Omega| = P(50, 5)$

$P(A) = \dfrac{1}{P(50,5)}$.

Theorems

1. $P(\emptyset) = 0$ $\quad \phi =$ impossible event

2. $P(\bar{A}) = 1 - P(A)$

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof 2: $\bar{A} = \Omega - A$ $\qquad |\bar{A}| = |\Omega - A| = |\Omega| - |A|$

$$P(\bar{A}) = \frac{|\bar{A}|}{|\Omega|} = \frac{|\Omega - A|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|} = \underbrace{\frac{|\Omega|}{|\Omega|}}_{1} - \underbrace{\frac{|A|}{|\Omega|}}_{P(A)}$$

$$= 1 - P(A)$$

Proof 3: $|A \cup B| = |A| + |B| - |A \cap B|$ (PIE)

$$P(A \cup B) = \frac{|A \cup B|}{|\Omega|} = \frac{|A| + |B| - |A \cap B|}{|\Omega|}$$

$$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|}$$

$$= P(A) + P(B) - P(A \cap B)$$

Example 5, Section 7.1

Find the probability that a hand of five cards in poker contain four of one kind.

# of 5-card hands $= \binom{52}{5} = |\Omega|$

A = the desired event

four of one kind — a fifth card

$\binom{13}{1}$ $\binom{48}{1}$ $= \binom{13}{1} \times \binom{48}{1}$

$\therefore P(A) = \dfrac{\binom{13}{1} \cdot \binom{48}{1}}{\binom{52}{5}}$

Ex8 A sequence of 10 bits are randomly generated. What is the prob. that $\underbrace{\text{one of the bits is a zero}}_{A}$?

$|\Omega| = 2^{10}$

$\bar{A}$ = Set of strings with all 1s $= \{1111111111\}$ $|\bar{A}| = 1$

$P(\bar{A}) = \dfrac{1}{2^{10}} = 2^{-10}$

$P(A) = 1 - P(\bar{A}) = 1 - \dfrac{1}{2^{10}} = \dfrac{1023}{1024}$

8': B = exactly one of the bits is zero. $|B| = \binom{10}{1} = 10$

$P(B) = \dfrac{10}{2^{10}}$.

Ex. 9

What is the prob. that a randomly selected positive int. ≤ 100 is divisible by 2 or 5?

Solution 1

$$n_2 = \left\lfloor \frac{100}{2} \right\rfloor = 50 \qquad n_5 = \left\lfloor \frac{100}{5} \right\rfloor = 20$$

$$LCM(2,5) = 10 \qquad n_{10} = \left\lfloor \frac{100}{10} \right\rfloor = 10$$

A = set of numbers divisible by 2 or 5

$$|A| = n_2 + n_5 - n_{10} = 50 + 20 - 10 = 60$$

$$P(A) = \frac{60}{100} = 0.6$$

Solution 2:

N_2 = event that a selected # is divisible by 2, $|N_2| = \left\lfloor \frac{100}{2} \right\rfloor = 50$

N_5 = " 5, $|N_5| = \left\lfloor \frac{100}{5} \right\rfloor = 20$

$$P(N_2 \cup N_5) = P(N_2) + P(N_5) - P(N_2 \cap N_5)$$

$N_2 \cap N_5$ = Divisible by 2 and 5 = Divisible by $LCM(2,5)$

$$|N_2 \cap N_5| = \left\lfloor \frac{100}{10} \right\rfloor = 10$$

$$P(N_2) = \frac{50}{100} = 0.5 \qquad P(N_5) = \frac{20}{100} = 0.2 \qquad P(N_2 \cap N_5) = \frac{10}{100} = 0.1$$

$$P(N_2 \cup N_5) = 0.5 + 0.2 - 0.1 = 0.6$$