Ω - Sample space of a RE

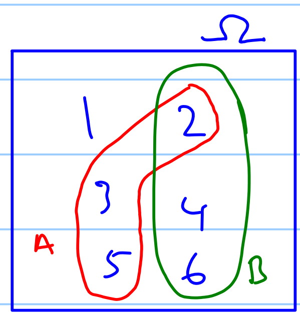
A, B are events on RE, $A, B \subseteq \Omega$.

Let $P(B) > 0$.

$P(A|B)$ = Prob. that A occurs given that B occurred.

$\Rightarrow$ re-evaluate the prob. of event A using B as the new sample space and $A \cap B$ as the new event space.

Example: RE = roll of a fair 6-face die. So the sample space Ω = {1, 2,394.56}.
Let A = prime number occurs = {2, 3.5}. P(A) = 3/6 = 1/2
and B = even number occurs = {2,4,6}. P(B) = 3/6 = 1/2.
In a roll of the die, it is observed that B occurred. What is P(A|B)?



If B occurred, then one of 2, 4, 6 must have occurred.

only 2 is the prime.

Once B occurs, new sample space is $\{2, 4, 6\}$

New event space, $A \cap B = \{2\}$

$$P(A|B) = 1/3 = \frac{|A \cap B|}{|B|} = \frac{|A \cap B| / |\Omega|}{|B| / |\Omega|} = \frac{P(A \cap B)}{P(B)}$$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}.}$$

$P(A \cap B) = \frac{1}{6}$ $\quad P(B) = \frac{3}{6}$

$$\frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = 1/3.$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$

Multiplication rule

$$P(A \cap B) = \begin{cases} P(A|B) \cdot P(B), & P(B) > 0 \\ P(B|A) \cdot P(A), & P(A) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(A \cap \overbrace{B \cap C}^{X}) = P(A \cap X) = P(X) \cdot P(A|X) = P(B \cap C) \cdot P(A|B \cap C)$$

$$P(A \cap B \cap C) = P(C) \cdot P(B|C) \cdot P(A|B \cap C)$$

In general,

If $A_1, A_2, \ldots, A_n$ are events and

$A_1 \cap A_2 \cap \cdots \cap A_{n-1} \neq \emptyset$, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

Example:

There are 100 computers, 75 of which are of brand x. If three computers are selected at random without replacement, what is the probability that each of the selected computers is of brand x.

A_i = event i^{th} selected computer is of brand x, $i = 1, 2,$ or 3

Need to calculate $P(A_1 \cap A_2 \cap A_3)$.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$$

$$P(A_1) = \frac{75}{100}$$

$$P(A_2 | A_1) = \frac{74}{99}, \quad \because \text{If } A_1 \text{ occurred, then only 74 'x' out of 99}$$

$$P(A_3 | A_1 \cap A_2) = \frac{73}{98}$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{75 \cdot 74 \cdot 73}{100 \cdot 99 \cdot 98}$$

Alternative solution using the counting technique.

$$p = \frac{\binom{75}{3} \leftarrow \text{event size}}{\binom{100}{3} \leftarrow \text{sample space size}}$$

$$= \frac{\frac{75 \cdot 74 \cdot 73}{3!}}{\frac{100 \cdot 99 \cdot 98}{3!}} = \frac{75 \cdot 74 \cdot 73}{100 \cdot 99 \cdot 98}$$

Independent Events

Two events A and B are independent if and only if $\boxed{P(A \cap B) = P(A) \cdot P(B)}$

In other words, P(A|B) = P(A), provided B is not a null event, and P(B|A) = P(B), provided A is a not null event.

If A and B are independent events, so are A' and B, A and B', and A' and B'.

In general, non-null events $A_1, A_2, \ldots, A_n$ are (statistically) independent if

$$\boxed{P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \cdots P(A_n)}$$

Example: Flip a fair coin repeatedly until a head occurs. If sequence of coin flips is of interest, then it is a RE.

$$\Omega = \{ \underset{\frac{1}{2}}{H}, \underset{\frac{1}{4}}{TH}, \underset{\frac{1}{8}}{TTH}, \underset{\frac{1}{16}}{TTTH}, \ldots \}$$

The probabilities of each sample point can be calculated by noting that each coin flip is independent of other flips and the product rule for occurrence of multiple independent events.

For a fair coin, $P(H) = P(T) = \frac{1}{2}$.

$$P(TH) = \underbrace{P(T)}_{\text{1st flip}} \cdot \underbrace{P(H|T)}_{\text{2nd flip}} = P(T) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

$$P(TTH) = P(T) \cdot P(T) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Note that $P(\Omega) = 1$ since the sum of probabilities of all sample points $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

$$\frac{1/2}{1-(1/2)} = 1$$

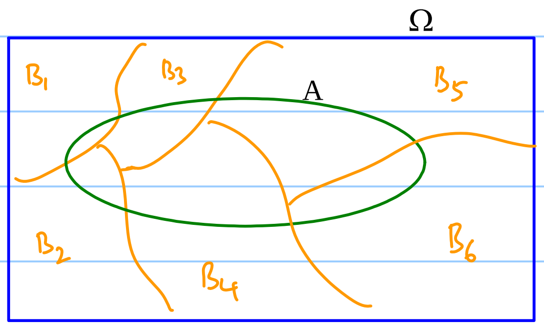
Law of Total Probability

Let $B_1, B_2, \ldots, B_n$ partition Ω.

(i) $B_i \neq \emptyset$, $1 \le i \le n$

(ii) $B_i \cap B_j = \emptyset$, $1 \le i < j \le n$

(iii) $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$



$P(B_1) + P(B_2) + \cdots + P(B_n) = 1$

Let A be another event

$P(A|B_1), P(A|B_2), \ldots, P(A|B_n)$ are known

as well as $P(B_1), \ldots, P(B_n)$ are known.

$P(A) = ?$

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$$

$$P(A) = P\big((A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)\big)$$

$$= P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$

$$\boxed{P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + \cdots + P(B_n) \cdot P(A|B_n)}$$

Example: A supercomputer center receives jobs from three different sources.

Source		% of Jobs	% of Jobs that require graphics processing	
1	$P(B_1)$	15%	1%	$\leftarrow P(A\|B_1)$
2	$P(B_2)$	35%	5%	$\leftarrow P(A\|B_2)$
3	$P(B_3)$	50%	2%	$\leftarrow P(A\|B_3)$

If a job is picked at random, what is the probability that it requires graphics processing?

Let B_i = event that the job picked @ random is from source i, $i = 1, 2$ or 3

Then B_1, B_2, B_3 partition Ω.

Let A = event that the job requires graphics processing

Then, By the Law of Total probability

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3)$$

$$= 0.15 * 0.01 + 0.35 * 0.05 + 0.5 * 0.02 = 0.029$$

If a job picked at random requires graphics processing, what is the prob. that it is from source 2? $P(B_2|A)$?

$$P(B_2|A) = \frac{P(B_2 \cap A)}{P(A)} = \frac{0.35 * 0.05}{0.029} = 0.603$$

$$P(B_2 \cap A) = P(B_2) \cdot P(A|B_2) = 0.35 * 0.05$$

Bayes Theorem

$$\boxed{P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i) \cdot P(A \mid B_i)}{P(A)},}$$

Where $P(A) = P(B_1) \cdot P(A \mid B_1) + \cdots + P(B_n) \cdot P(A \mid B_n)$

$B_i, \ldots, B_n$ are partitioning events and A is another event of interest.

Problem 6, section 7.3

The players of a soccer league are tested for drugs using a special test. With this test, 98% of players taking steroids test positive, and 12% of players not taking steroids test positive. It is estimated that 5% of all players take steroids . What is the probability that a player who tests positive takes steroids?

$\Omega = \{$ soccer players that are subjected testing $\}$

Let C = player is cheating (taking steroids)

$C = \{$soccer players taking steroids$\}$ $P(C) = 0.05$

$\bar{C} = \{$ soccer players not taking steroids$\}$ $P(\bar{C}) = 0.95$

C and $\bar{C}$ partition Ω.

T = player tests positive in a drug test

$P(T|C) = 0.98$ $P(T|\bar{C}) = 0.12$ Calculate $P(C|T)$.

$$P(T) = P(C) \cdot P(T/C) + P(\bar{C}) \cdot P(T/\bar{C}) = 0.05 * 0.98 + 0.95 * 0.12 = 0.163$$

$$P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{P(C) \cdot P(T|C)}{P(T)} = \frac{0.05 * 0.98}{0.163}$$

$$= 0.3006$$