Random Variables

4/11/12

A random variable (RV) is a real-valued function defined on the sample space Ω of a RE.

Let X be a RV defined on the sample space Ω. If ω ∈ Ω, then X(ω) is a real number.

Example: Consider the RE of 3 fair coin flips.

$$\Omega = \{\overset{1/8}{HHH},\ \overset{1/8}{HHT},\ \overset{1/8}{HTH},\ HTT,\ THH,\ THT,\ TTH,\ \overset{1/8}{TTT}\}$$

Let X be a RV denoting the # of heads observed.

$X(HHH) = 3 \qquad X(HHT) = 2 = X(HTH) = X(THH)$

$X(HTT) = 1 = X(THT) = X(TTH) \qquad X(TTT) = 0$

X partitions Ω   X takes 0, 1, 2, or 3

$P(X = a)$ = prob. X takes value a

$$\underbrace{P(X=3) = \tfrac{1}{8} \qquad P(X=2) = \tfrac{3}{8} \qquad P(X=1) = \tfrac{3}{8} \qquad P(X=0) = \tfrac{1}{8}}_{\text{prob. distribution of } X}$$

$$P(X \geq 2) = P(X = 2 \text{ or } X = 3) = P(X=2) + P(X=3)$$
$$= \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

$P(X=4) = 0 \qquad P(X=2.5) = 0$

$P(X=-1) = 0 \qquad P(-100) = 0$

P(X>=0) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = 1/8 + 3/8 + 3/8 + 1/8 = 1

Example:

Two players play a game by flipping a fair coin one or two times. If the first flip results in a head, then the game stops. Otherwise, the coin is flipped just one more time, and the game stops regardless of the outcome. If the game ends in a head, then player 1 loses \$1 to player 2. Otherwise, player 1 wins \$3 from player 2.

Give the sample space. If X is a RV denoting the winnings by player 1, give the values X takes.

$$\Omega = \{H, TH, TT\}$$

$$X(H) = -1$$

$$X(TH) = -1$$

$$X(TT) = 3$$

$$P(H) = \frac{1}{2}$$

$$\begin{aligned} P(TH) &= P(T \cap H) \\ &= P(T) \cdot P(H|T) \\ &= P(T) \cdot P(H) \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(TT) &= P(T) \cdot P(T) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Bernoulli trials

A RE that has exactly two outcomes — success with prob. p / failure with prob. 1-p.

is a Bernoulli trial.

1. Coin flip - Head (success) with prob. p

2. if (B) then { } else { }

P(success) + P(failure) = p + (1-p) = 1

Fair coin: P(heads) = p = 1/2.

Discrete uniform probability distribution

Consider a RE with finite sample space, Ω, such that |Ω| = n.

If each sample pooint is equally likely, then the probability distribution is uniform.

Each sample point occurs with probability 1/n.

Binomial probability distribution

Consider a sequence of **n** independent Bernoulli trials, each with **p** as the probability of success.

Let **X** = the number of heads observed in **n** coin flips

Then,

P(exactly k out of n flips resulted in heads) = $\mathbf{P(X=k) = \overbrace{C(n,k)}^{\#\text{ways k heads can occur in n flips}}\, \underbrace{p^k\,(1-p)^{n-k}}_{\text{prob. of one such scenario}}}$

$$\overbrace{\underline{H}\ \underline{H}\ \cdots\ \underline{H}}^{k}\ \overbrace{\underline{T}\ \cdots\ \underline{T}\ \underline{T}}^{n-k}$$

$$p * p * \cdots * p * (1-p) * \cdots * (1-p) = p^k (1-p)^{n-k}$$

indep. events

$$P(H_1 \cap H_2 \cap \cdots \cap H_k \cap T_{k+1} \cap \cdots \cap T_n)$$

$$= P(H_1)\cdot P(H_2)\cdots P(H_k)\cdot P(T_{k+1})\cdots P(T_n)$$

$$= \underbrace{p\ \ p\ \cdots\ p}_{k}\ \underbrace{(1-p)\cdots(1-p)}_{n-k}$$

$$= p^k (1-p)^{n-k}$$

A coin flip: p (head), $\underbrace{1-p}_{q}$ (tail)

$p + q = 1$.

$$\underbrace{(p+q)\cdot(p+q)\cdots(p+q)}_{n \text{ times}} = (p+q)^n$$

$$\boxed{\begin{aligned} &\underline{\text{B.T.}}\\ &(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1 y^{n-1}\\ &\quad + \binom{n}{2}x^2y^{n-2} + \cdots\\ &\quad + \binom{n}{n}x^n y^{n-n}\end{aligned}}$$

$$(p+q)^n = \binom{n}{0}p^0q^n + \binom{n}{1}p^1q^{n-1} + \binom{n}{2}p^2q^{n-2} + \cdots + \binom{n}{n}p^n\cdot q^{n-n}$$

$$\downarrow$$

$$1^n$$

$$\downarrow$$

$$1 = \underbrace{\binom{n}{0}q^n}_{P(X=0)} + \underbrace{\binom{n}{1}p\,q^{n-1}}_{P(X=1)} + \cdots + \underbrace{\binom{n}{k}p^kq^{n-k}}_{P(X=k)} + \cdots + \underbrace{\binom{n}{n}p^n}_{P(X=n)}$$

Discrete Random Variables

Let X defined on a Ω take values $x_1, x_2, \ldots, x_n$,

$x_1 \leq x_2 \leq \cdots \leq x_n$.

$P(X = x_1) = p(x_1)$ is the prob. that X takes value x_1.

$p(x_1)$: point prob.

$p(x_1), p(x_2), \ldots, p(x_n)$ define a prob. mass function (pmf) for RV X.

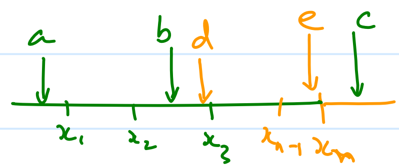
$\underbrace{P(X \leq x_i)}_{F(x_i)}$ = prob. that X takes values $x_1, x_2, \ldots,$ or x_i. $1 \leq i \leq n$.

$F(x_i)$ ← cumulative distribution function (CDF).

$$F(x_i) = P(X \leq x_i) = P(X = x_1) + P(X = x_2) + \cdots + P(X = x_i)$$
$$= p(x_1) + p(x_2) + \cdots + p(x_i) = \sum_{k=1}^{i} p(x_k).$$

More specifically, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

$$F(x_3) = p(x_1) + p(x_2) + p(x_3) = F(x_2) + p(x_3)$$
$$F(x_i) = F(x_{i-1}) + p(x_i), \quad i = 1, \ldots, n$$



$F(a) = 0$ if $a < x_1$

$F(b) = P(X \leq b) = p(x_1) + p(x_2)$

$F(c) = P(X \leq c) = P(X \leq x_n) = p(x_1) + \cdots + p(x_n) = 1$

$P(X \leq d) = P(X \leq x_1) + P(X \leq x_2)$

$F(e) = P(X \leq e) = P(X \leq x_{n-1}) = p(x_1) + p(x_2) + \cdots + p(x_{n-1}) = 1 - p(x_n)$

Example: Consider the RE of 3 fair coin flips.

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Let X be a RV denoting the # of heads observed.

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$X(HTT) = 1 = X(THT) = X(TTH) \qquad X(TTT) = 0$

<u>pmf:</u> $p(0) = P(X=0) = \frac{1}{8}, \quad p(1) = \frac{3}{8}, \quad p(2) = \frac{3}{8}, \quad p(3) = \frac{1}{8}$

$F(0) = p(0) = \frac{1}{8}, \quad F(1) = F(0) + p(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

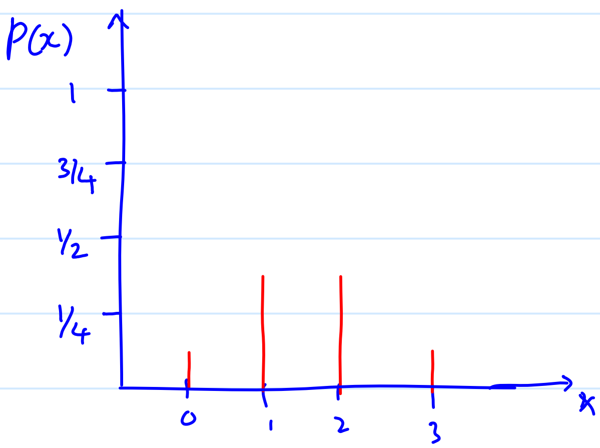
$F(2) = F(1) + p(2) = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$

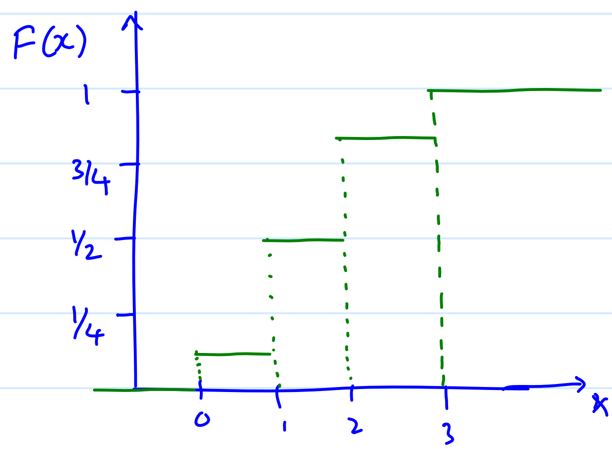
$F(3) = F(2) + p(3) = \frac{7}{8} + \frac{1}{8} = 1$

$F(4) = F(3) + \underset{0}{\underset{\shortparallel}{p(4)}} = 1 + 0 = 1$

If $a < 0$
$F(a) = 0$

If $b > 3$
$F(b) = 1$





Properties of pmf, p()

Let T = image of $X = \{x_1, x_2, \dots\}$

(i) $p(x_i) \geq 0 \ \forall$ all x_i

(ii) T is finite or countably infinite

(iii) $\sum_{x_i \in T} p(x_i) = 1$

Properties of CDF, F()

(i) F is a nondecreasing function.

$\Rightarrow$ If $x < y$, then $F(x) \leq F(y)$

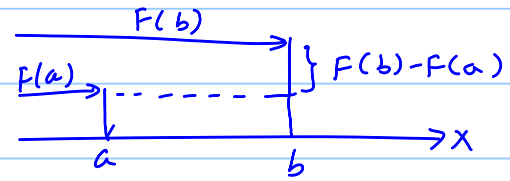
(ii) $\lim_{x \to +\infty} F(x) = 1$

(iii) $\lim_{x \to -\infty} F(x) = 0$

Let $a < b$. $\boxed{P(a < X \leq b) = F(b) - F(a)}$

$$F(b) = \underline{P(a < X \leq b)} + \underbrace{P(X \leq a)}_{F(a)}$$

$$\therefore F(b) - F(a) = P(a < X \leq b)$$



Expectation and Variance

The pmf and CDF describe a RV completely. The expectation and variance describe the RV concisely.

Expectation is denoted E(X) or μ. Variance is denoted V(X) or σ^2.

Let RV X take values $x_1, x_2, \ldots, x_n$ with nonzero probability.

$$\mu = E(X) = \sum_{k=1}^{n} x_k \, p(x_k)$$

Special case:

$$p(x_1) = p(x_2) = \cdots = p(x_n) = \frac{1}{n}$$

$$E(X) = \frac{1}{n} x_1 + \frac{1}{n} x_2 + \cdots + \frac{1}{n} x_n$$

$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\sigma^2 = V(X) = \sum_{k=1}^{n} (x_k - \mu)^2 \cdot p(x_k)$$

standard deviation, $\sigma = \sqrt{V(X)}$

Coefficient of variation $CV = \frac{\sigma}{\mu}$

Example: Let X be the RV indicating the number of heads in a 3 fair-coin flop RE.

$p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$

$$\mu = E(X) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$

$$= 0 + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3+6+3}{8} = \frac{12}{8} = 1.5$$

$$V(X) = (0-1.5)^2 \cdot p(0) + (1-1.5)^2 \cdot p(1) + (2-1.5)^2 \cdot p(2) + (3-1.5)^2 \cdot p(3)$$

$$= 0.75$$

$\sigma = \sqrt{0.75}$ $\qquad CV = \frac{\sigma}{\mu} = \frac{\sqrt{0.75}}{1.5}$