Problems on Random Variables 4/23/12

Note Title

Problem on discrete RVs: (similar to 2.40, 2.43, and 2.46 in [S3])

X		2	3
$P_x(\ )$	0.6	0.3	0.1
$F(x)$	0.6	0.9	1.0

$$\mu_x = E(X) = 1*0.6 + 2*0.3 + 3*0.1$$
$$= 0.6 + 0.6 + 0.3 = 1.5$$

$$\underline{E(x^2)} = \sum x^2 p(x) = 1^2*0.6 + 2^2*0.3 + 3^2*0.1$$
$$= 0.6 + 4*0.3 + 9*0.1 = 0.6 + 1.2 + 0.9 = 2.7$$

$$V(X) = \sum (x-\mu)^2 p(x) = (1-1.5)^2 \cdot 0.6 + (2-1.5)^2 * 0.3 + (3-1.5)^2 * 0.1$$
$$= 0.5^2 * 0.6 + 0.5^2 * 0.3 + 1.5^2 * 0.1 = \underline{0.45}$$

$$V(X) = E(X^2) - [E(X)]^2 = 2.7 - 1.5^2 = 2.7 - 2.25 = 0.45$$

$$E(2X) = 2\,E(X) = 2*1.5 = 3$$

$$V(3X) = 3^2\,V(X) = 9*0.45 = 4.05$$

Coeff. of Variation $C_x = \dfrac{\sigma_x}{\mu_x} = \dfrac{\sqrt{0.45}}{1.5} = \dfrac{0.6704}{1.5} = 0.447$

2.94 P.67

a

X \ Y	0	1	2	Σ
0	1/18	1/9	1/6	$= \frac{1}{18}+\frac{1}{9}+\frac{1}{6} = \frac{1+2+3}{18} = \frac{6}{18} = \frac{1}{3} = P(X=0) = P_X(0)$
1	1/9	1/18	1/9	$= \frac{1}{9}+\frac{1}{18}+\frac{1}{9} = \frac{2+1+2}{18} = \frac{5}{18} = P_X(1)$
2	1/6	1/6	1/18	$= \frac{1}{6}+\frac{1}{6}+\frac{1}{18} = \frac{3+3+1}{18} = \frac{7}{18} = P_X(2)$
	$\frac{1}{18}+\frac{1}{9}+\frac{1}{6} = \frac{6}{18}$ $P_Y(0)$	$\Sigma = \frac{6}{18}$ $P_Y(1)$	$\Sigma = \frac{6}{18}$ $P_Y(2)$	

b. Find $P(1 \le X < 3,\ Y \ge 1) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} + \frac{1}{18}$

$$= \frac{1+2+3+1}{18} = \frac{7}{18}$$

c. Determine whether X and Y are independent? No

$$P(X=k_1, Y=k_2) = P(X=k_1)\cdot P(Y=k_2)$$

$$P(X=0, Y=0) = 1/18$$

$$P(X=0)\cdot P(Y=0) = \frac{6}{18}\cdot\frac{6}{18} = \frac{1}{9} \neq P(X=0, Y=0)$$

2.55(c) to show X and Y are indep. $\boxed{P(X=x, Y=y) = cxy}$

$$P(X=k_1, Y=k_2) = c \cdot k_1 k_2$$

$$P(X=k_1) \cdot P(X=k_2) =$$

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

$$0 < r < 1$$

2.81 $P(X=x) = \frac{c}{3^x}$ (pmf) $\quad x = 1, 2, \cdots, \infty$

calculate c.

$$\sum_{i=1}^{\infty} \frac{c}{3^i} = 1 \qquad c\Big[\underbrace{\overset{a}{\frac{1}{3}} + \frac{1}{3^2} + \cdots}_{}\Big] = 1 \quad r = 1/3$$

$$c \cdot \frac{1/3}{1 - 1/3} = 1$$

$$c \cdot \frac{1/3}{2/3} = 1 \quad \text{or} \quad c = 2$$

$$\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^k} = \frac{\frac{1}{3} - \frac{1}{3^{k+1}}}{1 - \frac{1}{3}} = \frac{1}{2} \cdot \left[1 - 3^{-k}\right]$$

$$E(a f(x) + b g(y)) = a E(f(x)) + b E(g(y))$$

$$E(ax + by) = aE(x) + bE(y)$$

3.55b

$$X = \begin{cases} 1 & \text{prob. } 1/3 \\ 0 & \text{prob } 2/3 \end{cases} \qquad Y = \begin{cases} 2 & \text{prob. } 3/4 \\ -3 & \text{prob } 1/4 \end{cases}$$

$$E(2x^2 - y^2)$$

$$E(y) = 2 \times \frac{3}{4} + (-3) \cdot \frac{1}{4} = \frac{3}{4}$$

$$E(x^2) = (1)^2 \cdot \frac{1}{3} + (0)^2 \cdot \frac{2}{3} = \frac{1}{3}$$

$$E(y^2) = (2)^2 \cdot \frac{3}{4} + (-3)^2 \cdot \frac{1}{4} = 3 + \frac{9}{4} = \frac{21}{4}$$

$$E(2x^2 - y^2) = 2E(x^2) - E(y^2)$$

$$= 2 \cdot \frac{1}{3} - \frac{21}{4} = \frac{-55}{12}$$

$E(x^2 y) = E(x^2) \cdot E(y)$ if x and y are indep.

$$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

3.46 Calculate the expected number of points in 3 successive tosses of a fair die.

1 toss of a die

$x \to$	1	2	3	4	5	6
$p(x=k)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{(6 \cdot 7)/2}{6} = \frac{7}{2} = 3.5$$

Let X_i be the number of points seen in the ith toss of the die.

$$E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 3.5 + 3.5 + 3.5 = 10.5$$

4.61 A fair coin is tossed 6 times

X = # of heads $\qquad X \sim \text{Binomial}(6, 1/2)$

$$P(X=5) = \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 = \binom{6}{5}\cdot 2^{-6} = \frac{6}{64}.$$

$P(X \le 3)$ = prob. that 3 or fewer heads observed

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \binom{6}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^5 + \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 + \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3$$

$$= 2^{-6} + 6\cdot 2^{-6} + 15 * 2^{-6} + 20 * 2^{-6}$$

$$= 42 * 2^{-6} = \frac{42}{64}.$$

$$= \sum_{k=0}^{3}\binom{6}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{6-k}$$

4.67 True-false exam $\qquad$ T/F are equally likely

X = # of correct guesses $\qquad n = 10$, p = prob. that an answer is guessed correctly

Prob. of guessing correctly at least 6 of 10 T-F Q's

$$P(X \ge 6) = \sum_{k=6}^{10}\binom{10}{k} p^k (1-p)^{10-k}$$

4.68. Hint:

$P(\text{success}) = P(\text{a person is alive after 30 yrs}) = \frac{2}{3}$

4.90 $P(\text{a bulb is defective}) = 0.03 = p$

Sample Size $= n = 100$ bulbs.

poisson paremeter $\alpha = np = 100 * 0.03 = 3$

(a) Prob. that exactly '0' of the 100 bulbs are defective

$$P(X=0) = \frac{\alpha^0}{0!} e^{-\alpha} = e^{-\alpha} = e^{-3} = 0.04979$$

$$P(X=0) = \binom{100}{0} 0.03^0 \, 0.97^{100} =$$

4.91 (a) Prob. of more than 5 bulbs defective

$$P(X>5) = P(X \geq 6) = 0.083918$$

$$= \sum_{k=6}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = 1 - \underbrace{P(X \leq 5)}_{P(X=0)+\cdots+P(X=5)}$$

4.93. Hint $\frac{3}{100000} = 0.00003 = P(\text{drowning})$

$n = 200000$ $\alpha = np = 200000 * \frac{3}{100000} = 6$

4.92 1 red and 7 white marbles

$P(\text{a red marble is drawn}) = \frac{1}{8} = p$

in 8 drawings $n=8$

$P(\text{red marble is drawn 3 times}) =$

Problem 11, p. 492 [KR] Geometric RV

Roll a die until a 6 comes up or rolled 10 times.

What is the expected # of rolls

$P(\text{a Six}) = p = \frac{1}{6}$ (S) $\qquad P(\text{no Six}) = \frac{5}{6}$ (F) q

$S = p$ $\qquad FFFS\ q^3p$ $\qquad F^6S$ $\qquad F^9S$

$FS = qp$ $\qquad F^4S$ $\qquad F^7S$ $\qquad F^{10}S$

$FFS\ q^2p$ $\qquad F^5S$ $\qquad F^8S = q^8p$ $\qquad F^{11}S$

$$1 - [p + qp + q^2p + \cdots + q^8p]$$

$$1 - [p(1+q+\cdots+q^8)] = 1 - p\cdot\frac{1-q^9}{1-q}$$

$$= 1 - (1-q^9) = q^9$$

Note: $p = 1-q$